

Subquadratic algorithms for the diameter and the sum of pairwise distances in planar graphs

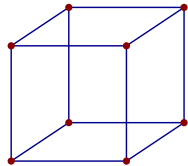
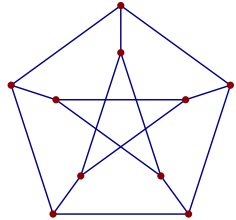
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SODA 2017
ACM TALG

Context

Basic parameters of a graph
(edge-weighted)

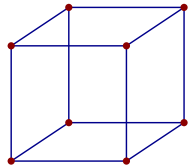
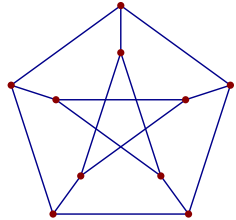
- ▶ number of vertices
- ▶ number of edges
- ▶ average/max/min degree
- ▶ diameter
- ▶ average distance / sum of distances
- ▶ girth
- ▶ connectivity
- ▶ ...



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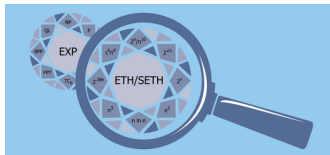
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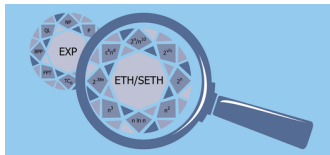
Context

Fine-grained complexity
Conditional lower bounds
based on a few *basic* assumptions



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[Roditty, Vassilevska Williams]

Computing the diameter of (unweighted) graphs with n vertices
and $O(n)$ edges in $O(n^{2-\delta_0})$ time



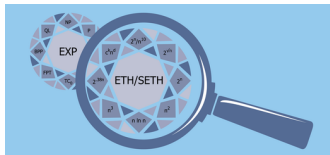
SAT with m variables can be solved in $O((2 - \delta_1)^m)$ time



Strong Exponential Time Hypothesis (SETH) is false

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Strong Exponential Time Hypothesis (SETH) is false

- ▶ Same result holds for *sum of distances*
- ▶ Find **graph classes** where the bound does not hold

Context

- ▶ Find **graph classes** where we can compute the diameter and the sum of distances in $O(n^{2-\delta_0})$ time
- ▶ Treewidth k in $O(n \text{ polylog } n)$ time
[C., Knauer; Abboud, Vassilevska Williams, Wang]

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Our result:

Diameter and sum of the pairwise distances in **planar** graphs with positive edge weights in $O(n^{11/6} \text{ polylog } n)$ expected time

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- ▶ Restricting attention to planar graphs is usual
- ▶ Best previous result for planar graphs: $O(n^2(\log \log n)^4 / \log n)$
[Wulff-Nilsen]

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[Wulff-Nilsen]
- ▶ Best next result for planar graphs: $\tilde{O}(n^{10/6})$
[Gawrychowski, Kaplan, Mozes, Sharir, Weimann]

How?

Graph Algorithms & Computational Geometry

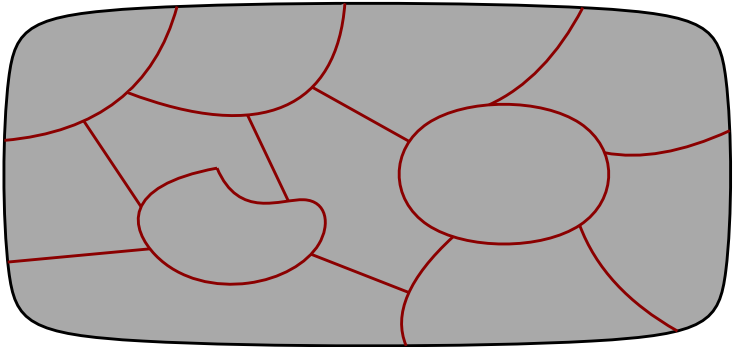
- ▶ r -division of planar graphs (includes separators)
- ▶ preprocessing of planar graphs for sum/max weight inside query subgraph
- ▶ dense-distance graphs (includes multiple-source shortest paths)
- ▶ (additively-weighted) Voronoi diagrams in planar/plane graphs
- ▶ abstract Voronoi diagrams
- ▶ randomized incremental constructions
- ▶ ...

Using tools by several researchers as black box.

Using r -divisions

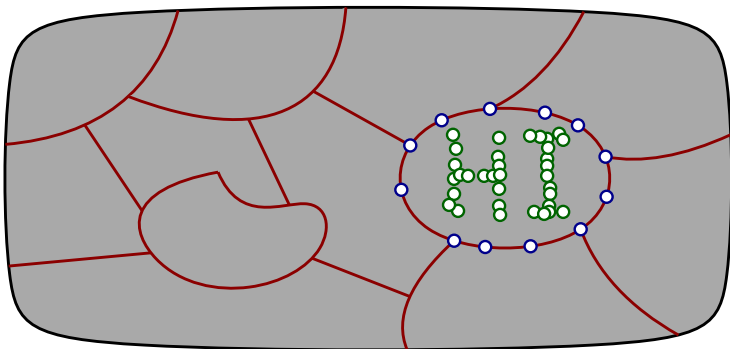
planar

Using r -divisions



$\Theta(n/r)$ pieces, $\Theta(r)$ vertex per piece

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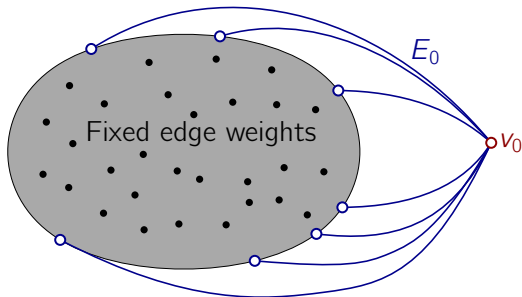


$\Theta(n/r)$ pieces, $\Theta(r)$ vertex per piece

$O(\sqrt{r})$ boundary vertices per piece

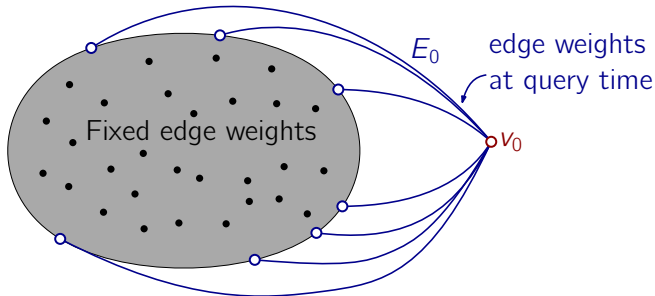
Main new ingredient

Data: planar graph H with distinguished vertex $v_0 \in V(H)$.
Weights for $E(H) \setminus E_0$ fixed, where E_0 edges of H incident to v_0 .



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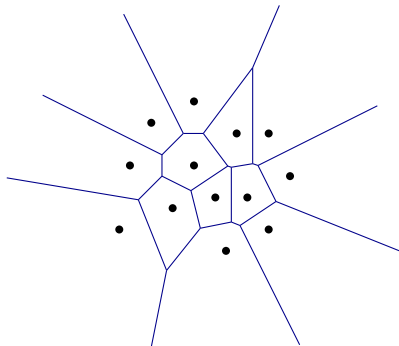
Preprocessing time: $O(b^3 r^2)$ time, where $r = |V(H)|$ and $b = |E_0|$

Query time: $O(b \text{ polylog } b)$ expected time

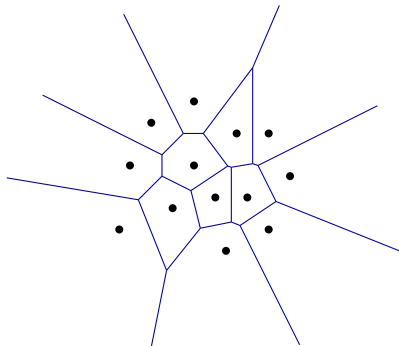
Using: Voronoi diagrams, randomized incremental construction

Preprocessing $O(r^{10})$ would also lead to subquadratic time.

Voronoi diagrams – Classical

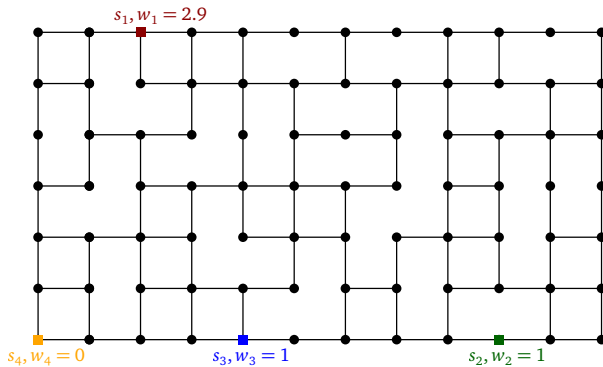


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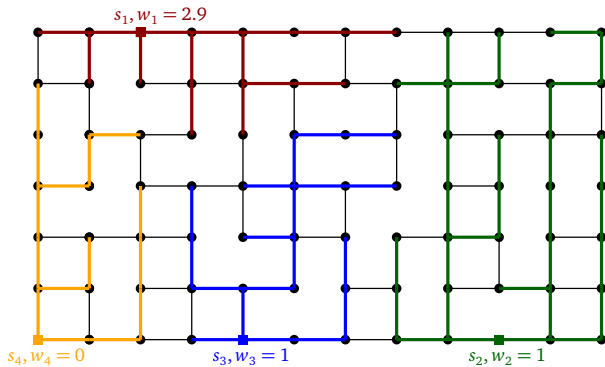


- ▶ Can be generalized to arbitrary metric spaces
- ▶ Additive-weights on the sites
- ▶ Abstract Voronoi diagrams: just needs “bisectors” in the plane and **some** other properties

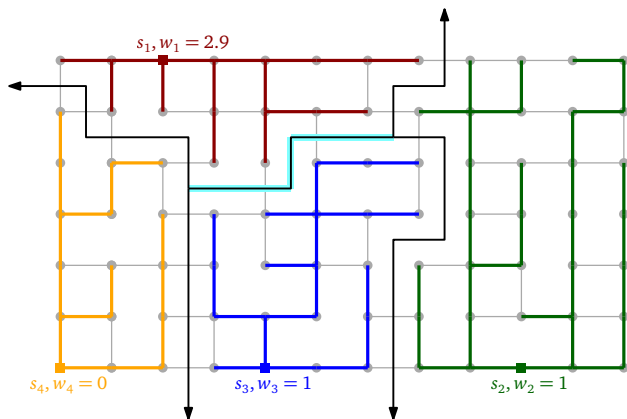
Additively-weighted VD in plane graphs



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Bisectors are curves defined by the dual graph
Edges of the VD described by 4 sites

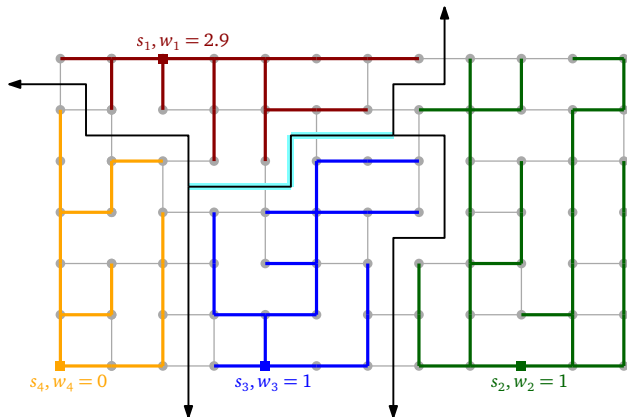
Additively-weighted VD in plane graphs

Via [Klein, Mehlhorn, Meiser '93] (randomized incremental construction):

- ▶ we can “compute” the additively-weighted VD in a plane graph
- ▶ all sites must be co-facial
- ▶ expected $O(s \log s)$ elementary operations, where s nb of sites
- ▶ computing the VD of 4 sites is an **elementary operation**
- ▶ the output is a combinatorial description of the VD in the dual graph

Independent of the size of the graph!!

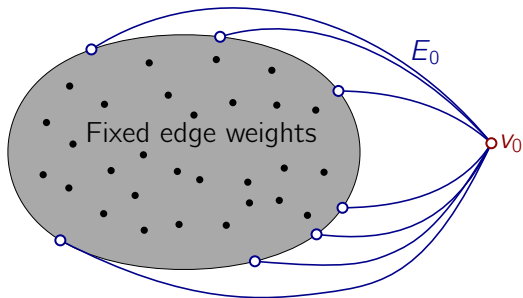
Description of the output



How to use this?

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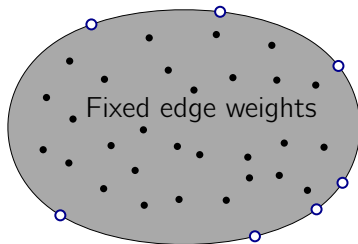
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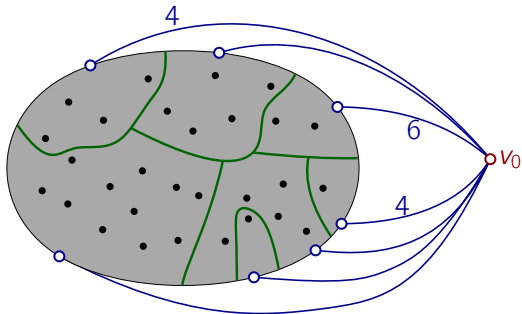
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Query time: weights at $E_0 \Rightarrow$ weights at sites
in $O(b \log b)$ time we get the Voronoi regions
have to “collect” the data from each region ... preprocessing

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- ▶ Follow up work: $\tilde{O}(n^{5/3})$ [Gawrychowski et al.]
- ▶ Follow up work: exact-distance oracle for planar graphs [Cohen-Addad et al.]

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- ▶ Sum of distances and diameter for H -minor-free graphs
- ▶ Computing n distances in planar graphs in $O(n^{(4/3)-\delta_0})$

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Many **THANKS** for your time!