Weighted k-Server Bounds via Combinatorial Dichotomies

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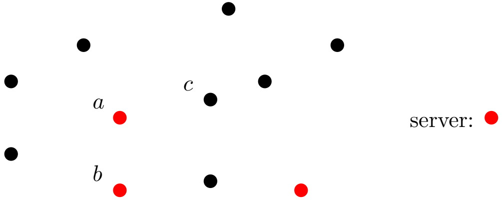
- Metric Space of n points, k servers.
- In each time step a point is requested.
- To serve the request, need to move one server there.



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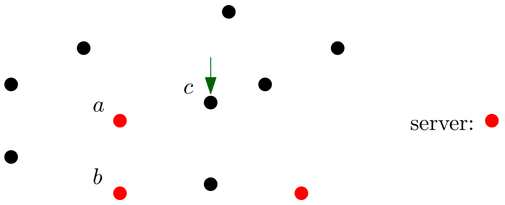

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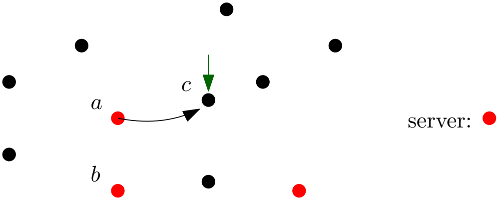

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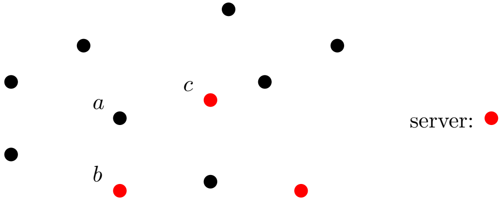

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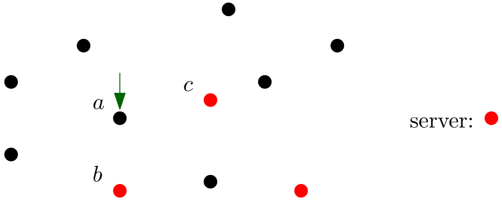

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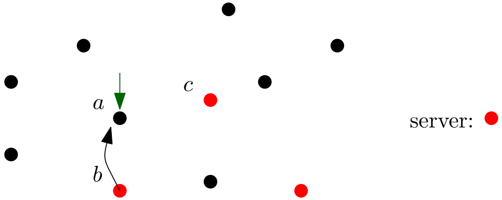

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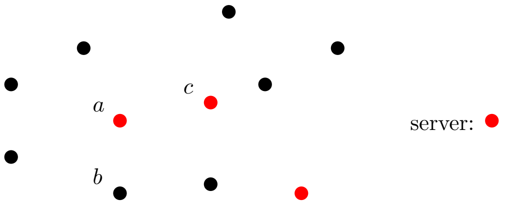

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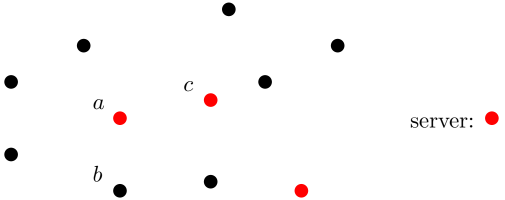

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Goal: Minimize total distance traveled by the servers.

Performance Measure: Competitive Ratio $= \max_{\sigma} \frac{ALG(\sigma)}{OPT(\sigma)}$

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Morally: Arbitrary n is not much harder than $n = k + 1$.

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- Qualitatively very different than (unweighted) k-server.
- Focus on uniform metrics (all distances 1)
  - Even this case not well understood

Weighted k-Server – Previous Work

Lower Bound: $(k+1)! - 1$ $(\approx 2^{k \log k})$ [Fiat, Ricklin '94]

Even for metrics of $n = k+1$ points

Upper Bounds: $2^{2^{O(k)}}$ [Fiat, Ricklin '94, Chiplunkar, Viswanathan '13]

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- True for $n = k+1$
  Connection with Metrical Service/Task Systems (MSS)
- True for $k = 2$
  - WFA is $(k+1)! - 1 = 5$-comp. [Chrobak, Sgall 2000]

Conjecture for $(k+1)! - 1$ Upper Bound

(unweighted) k-Server

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4. Goal: Show that WFA is $O((k+1)!)$-comp.

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3. **For d distinct weights, generalized WFA is $2^{O(d \cdot k^{d+3})}$-competitive**
   - This corresponds to paging with d cache types
   - For $d = O(1)$, this gives $2^{\text{poly}(k)}$

Open Problems

Randomized Algorithms?

- Best known $2^{2^{k+O(1)}}$, lower bound $\Omega(\log k)$.
- Natural to guess $2^{O(k)}$ bound

Weighted k-server on other metrics? (line, trees, arbitrary)

- No $f(k)$-comp. for $k > 2$. Gen. WFA the only candidate.

Generalized k-server:

- Server s_i has its own metric M_i.
- Request:$(r_1, r_2, \ldots, r_k)$. Need to move some s_i to r_i.
- k-server: all metrics same $M_i = M$, request $(r, r, \ldots, r)$
- Weighted k-server: $M_i = w_i \cdot M$, request $(r, r, \ldots, r)$
- $O(1)$-competitive for $k = 2$ [Sitters, Stougie '06, Sitters '14]
- No $f(k)$-comp. known for $k > 3$.

THANK YOU!