The Power of Vertex Sparsifiers in Dynamic Graph Algorithms

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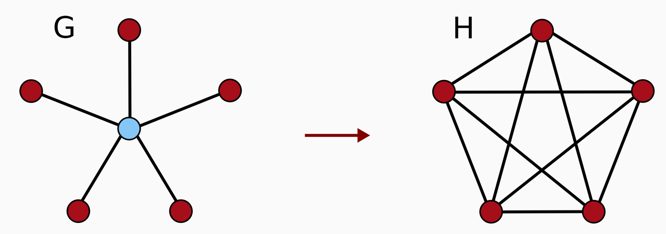
Vertex Sparsifier

- $G = (V, E)$ undirected, unweighted graph, $n = |V|$, $m = |E|$
- **Laplacian:** $L = D_G - A_G$, **Pseudo-inverse** $L^{\dagger}$
- The **effective resistance** between u and v in G is

$$R_G(u, v) := (1_u - 1_v)^{\intercal} L^{\dagger} (1_u - 1_v)$$

- Given G with k **terminals** $K \subset V(G)$, $H = (V_H, E_H, w_H)$ with $K \subset V_H$ is a α-**vertex resistance sparsifier** of G if for all t, t' from K:

$$\alpha \cdot R_G(t, t') \leq R_H(t, t') \leq R_G(t, t')$$



- (**Fast Vertex Sparsifier** [4]) There is a $\tilde{\mathcal{O}}(m\varepsilon^{-2})$ algorithm to compute a vertex resistance sparsifier H with

$$\alpha = (1 - \varepsilon) \text{ and } |E_H| = \tilde{\mathcal{O}}(k/\varepsilon^2)$$

Dynamic Effective Resistance

- Build data-structure that supports the following operations:
- **Insert**(u, v): insert the edge (u, v) in G
- **Delete**(u, v): delete the edge (u, v) from G
- **Query**(s, t): retun (approximate) $R_G(s, t)$

Motivation:

- natural, fundamental problem
- lots of recent work in the static setting e.g., fast laplacian solvers
- can it help us solve dynamic max-flow?

Our Result [1]

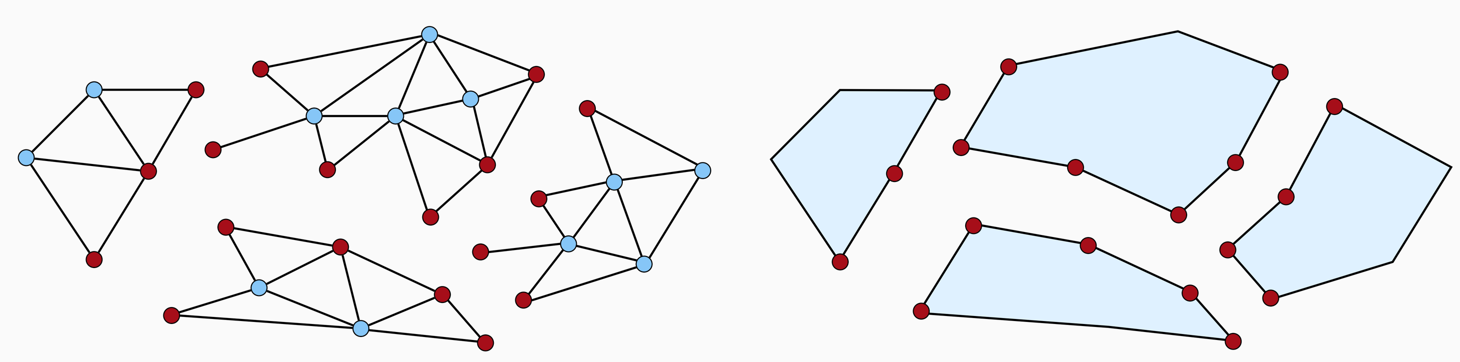
Theorem 1: When G is **planar**, there is a fully-dynamic algorithm that $(1 - \varepsilon)$-approximates effective resistances with

- $\tilde{\mathcal{O}}(n^{2/3}\varepsilon^{-2})$ update and query time

Idea: [Graph Clustering] + [Vertex Sparsification]

Data Structure

- partition G into n/r clusters G_i, each of size $\mathcal{O}(r)$
- each G_i has at most $\mathcal{O}(\sqrt{r})$ boundary nodes K_i
- compute a vertex sparsifiers H_i for each G_i w.r.t. K_i
- **cost:** $\tilde{\mathcal{O}}\left(n + (n/r) \cdot r\varepsilon^{-2}\right) = \tilde{\mathcal{O}}(n\varepsilon^{-2})$
- re-compute every $\mathcal{O}(n/r)$ operations



Handling Updates and Queries

Insert(u, v)

- declare u and v **boundary** in the clusters G_u and G_v
- re-compute from scratch sparsifiers H_u and H_v
- set (u, v) to be a **new** cluster
- **cost:** $\tilde{\mathcal{O}}(r\varepsilon^{-2})$

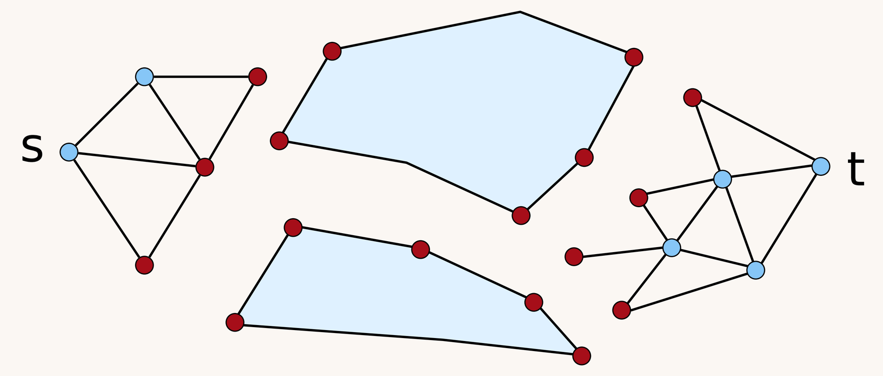
Delete(u, v)

- recompute from scratch a sparsifier $H_{(u,v)}$ of $G_{(u,v)}$
- **cost:** $\tilde{\mathcal{O}}(r\varepsilon^{-2})$

Query(s, t)

- let G_s and G_t the clusters containing s and t
- define **auxiliary** $H = G_s \cup G_t \cup \bigcup_{i \neq s,t} H_i$
- compute $R_H(s, t)$ using a Laplacian solver
- **cost:** $\tilde{\mathcal{O}}\left((r + n/\sqrt{r})\varepsilon^{-2}\right)$

Claim: $R_H(s, t)$ is a $(1 - \varepsilon)$–approx. to $R_G(s, t)$



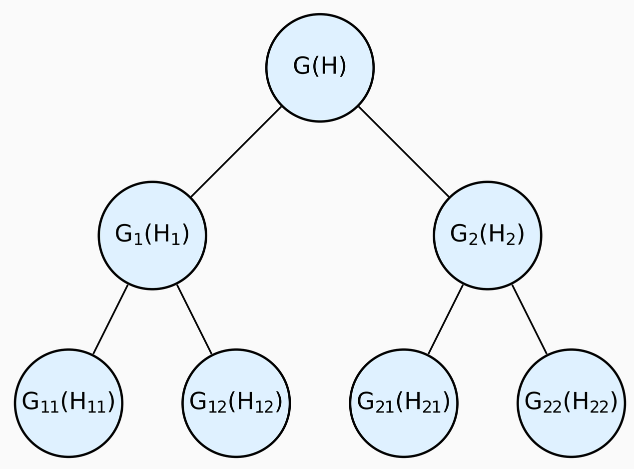
More Results [2]

Theorem 2: For **planar** graphs, can get the following improvement:

- $\tilde{\mathcal{O}}(n^{1/2}\varepsilon^{-2})$ update and query time.

Idea:

- Nested Dissection + Vertex Spars.



Theorem 3: Assuming **OMv** conjecture, in **general** graphs no dynamic algorithm can maintain **exact** $R_G(s, t)$ with

- $\mathcal{O}(n^{1-\delta})$ update and $\mathcal{O}(n^{2-\delta})$ query time

Recent Development [3]

For **general** graphs, there is a fully-dynamic algorithm that $(1 - \varepsilon)$-approximates effective resistances with

- $\tilde{\mathcal{O}}(m^{4/5}\varepsilon^{-4})$ update and query time.

References

[1] G. Goranci, M. Henzinger, and P. Peng. The Power of Vertex Sparsifiers in Dynamic Graph Algorithms. ESA 2017.
[2] G. Goranci, M. Henzinger, and P. Peng. Dynamic Effective Resistances and Approximate Schur Complement on Separable Graphs. ArXiv 2018.
[3] D. Durfee, Y. Gao, G. Goranci and R. Peng. Fully Dynamic Effective Resistances. ArXiv 2018.
[4] D. Durfee, R. Kyng, J. Peebles, A. B. Rao and S. Sachdeva. Sampling random spanning trees faster than matrix multiplication. STOC 2017.

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