# Online bipartite matching with $O(\log^2 n)$ replacements.

Aaron Bernstein – Technical University of Berlin
Jacob Holm – University of Copenhagen
Eva Rotenberg – University of Copenhagen

# Standard Online Matching

Bipartite Graph

- right side: **servers** are fixed from the beginning
- left side: **clients** arrive one a time with all incident edges

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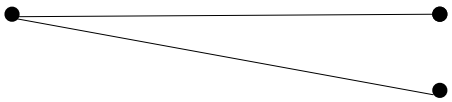

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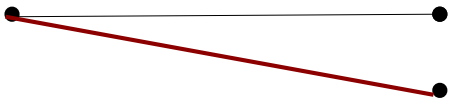

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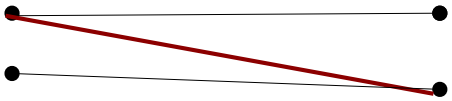

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Mistakes are unavoidable: must settle for an approximation

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## Online Matching With Recourse

- Can change the matching as new clients arrive
- Must always maintain an *exact* maximum matching
- **Goal:** Minimize total number of changes

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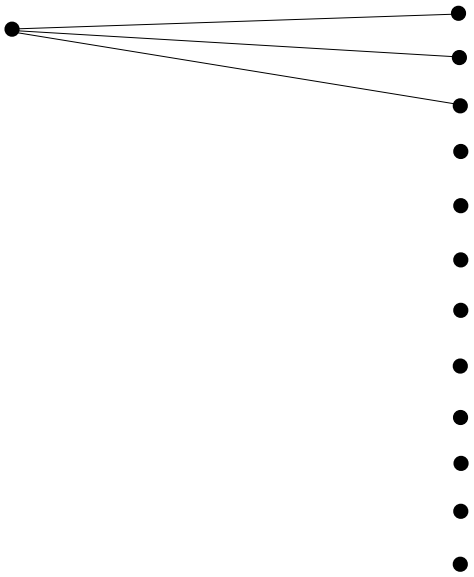

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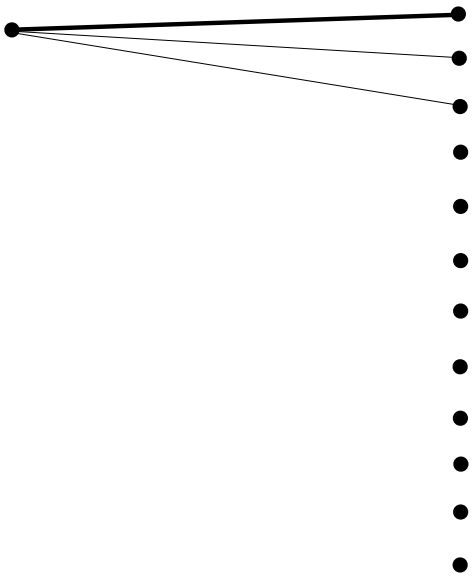

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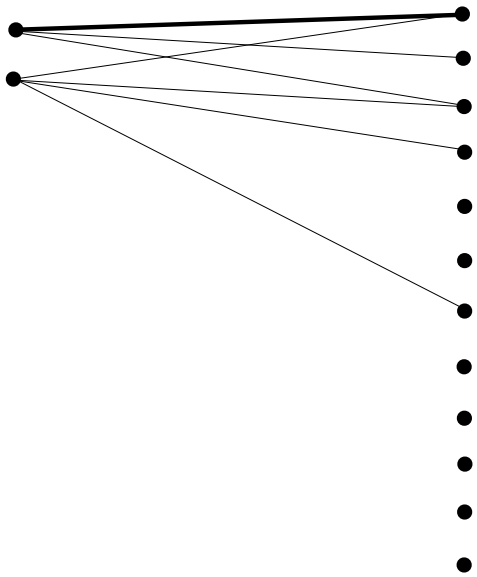

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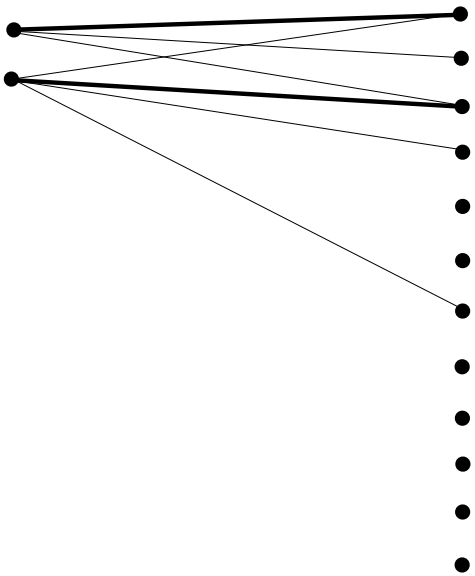

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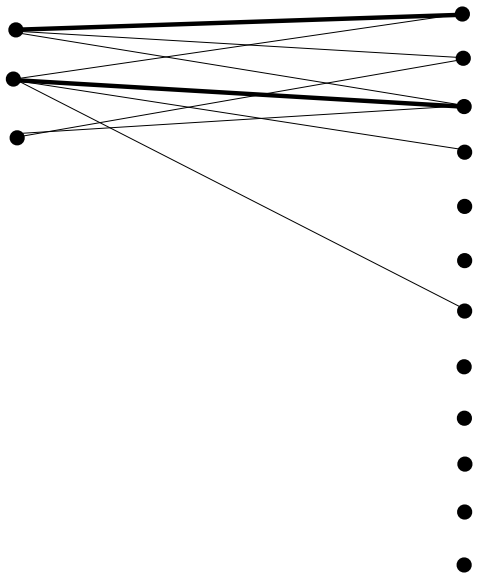

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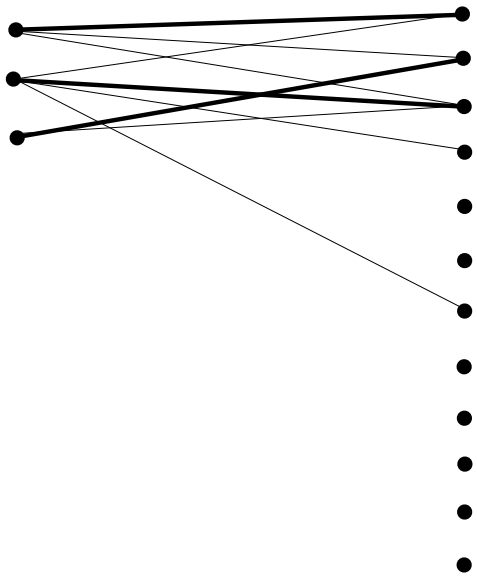

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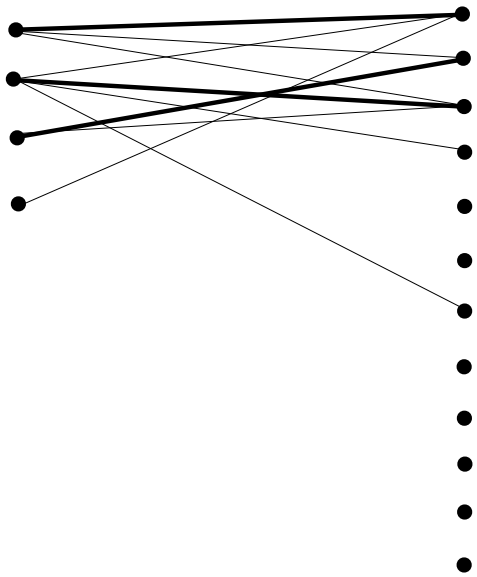

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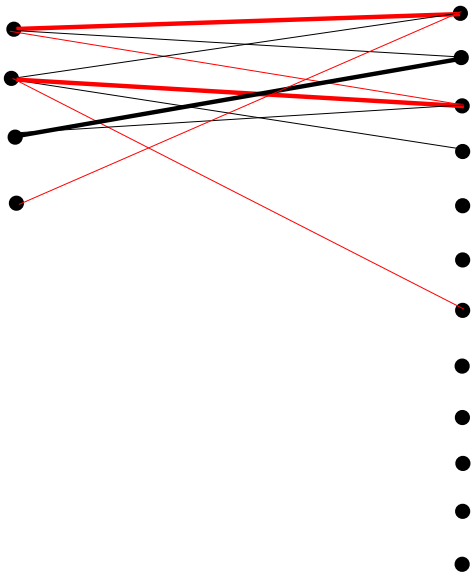

Must reassign 2 clients

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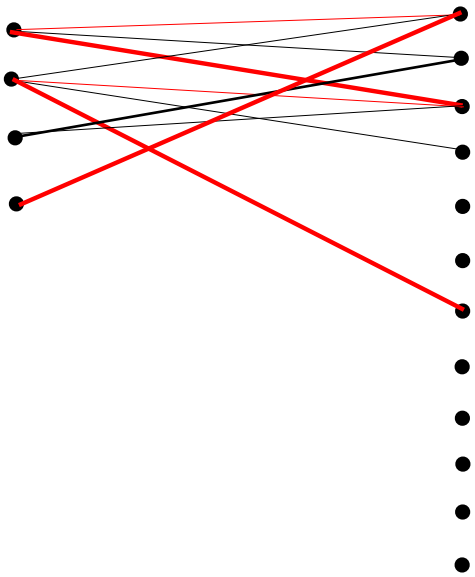

## Task

- Find an augmenting path for each arriving client
- **Goal:** minimize total length of augmenting paths

# With and Without Recourse: Comparison

## Online Matching Without Recourse

- never reassign clients
- **Goal:** good approximation
  - Deterministic: .5-approx.
  - Randomized: .63 approx. [Karp, Vazirani, Vazirani, '90]

## Online Matching With Recourse

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Other Problems with Recourse

- MST, Max Flow, Scheduling, Steiner Tree, Set Cover
- **goal:** bound how often the optimal solution changes

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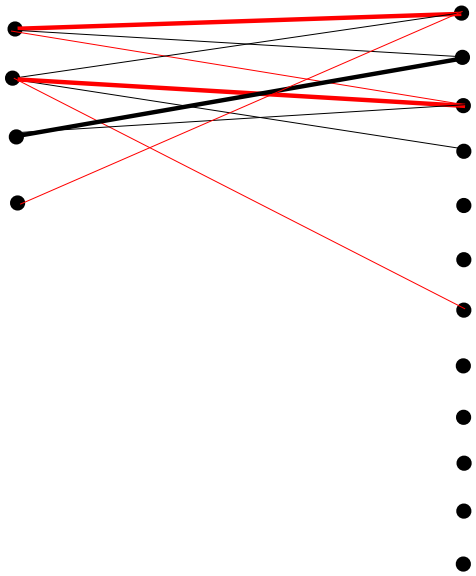

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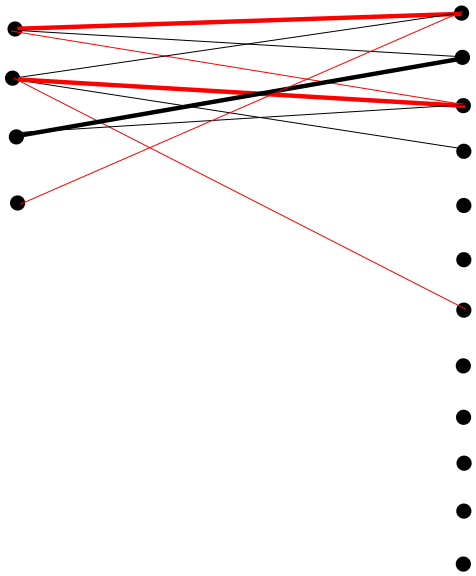

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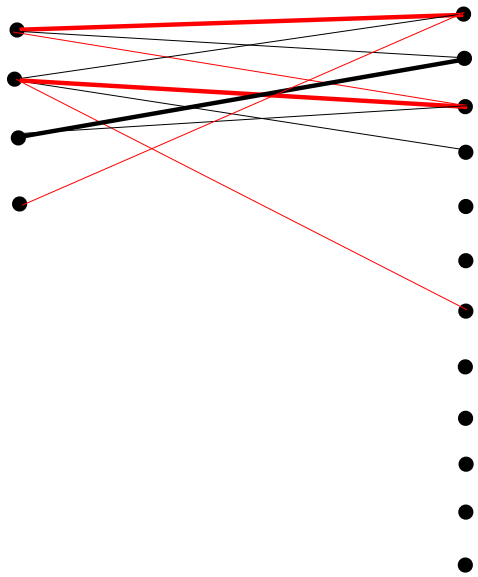

**Conjecture (2009):** SAP is asymptotically optimal

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## Our Result

SAP: $O(n \log^2 n)$ total number of changes ($\log^2 n$ amortized)

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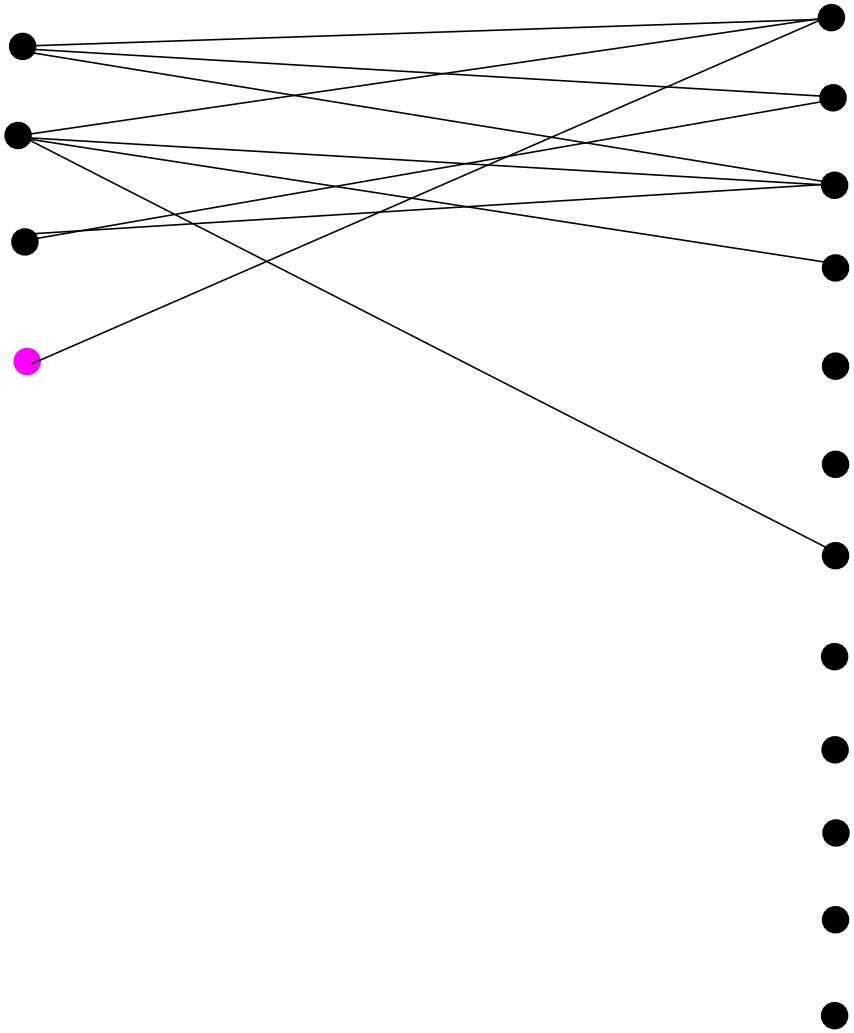
## Our Result: Matching Oblivious

- Does not need to "remember" matching between steps.
- Charges changes to potential $\Phi(G)$.

# Oblivious Augmenting Paths



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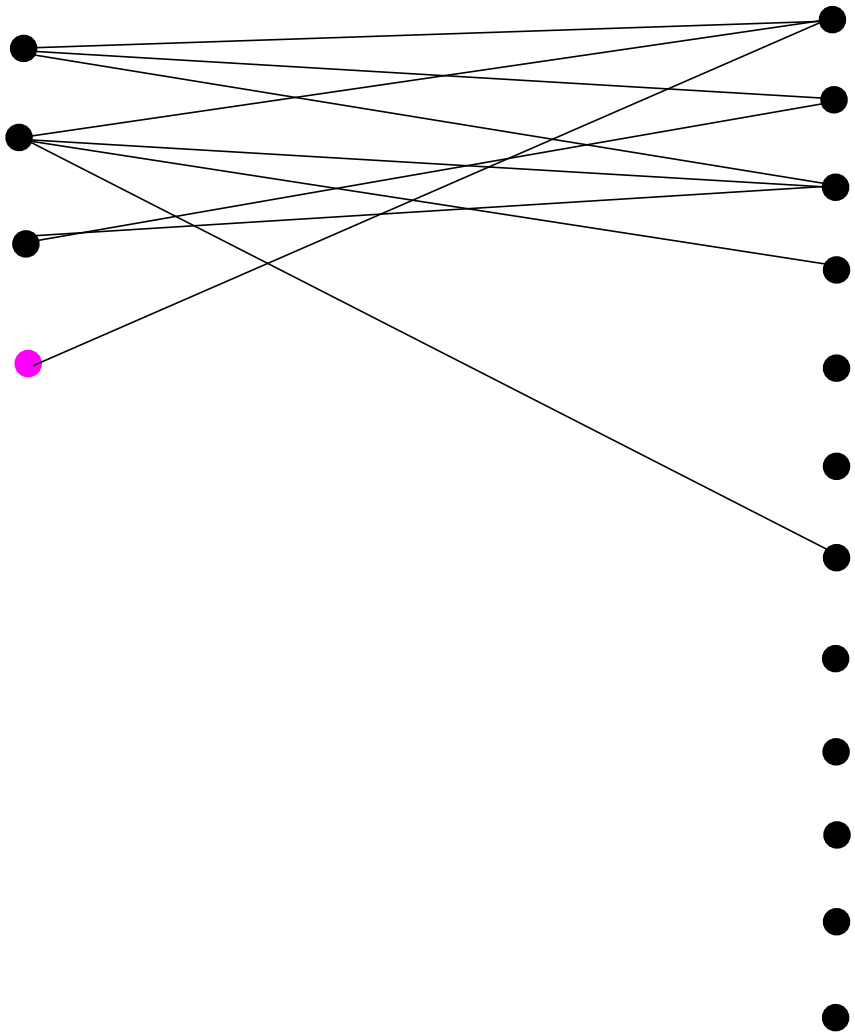

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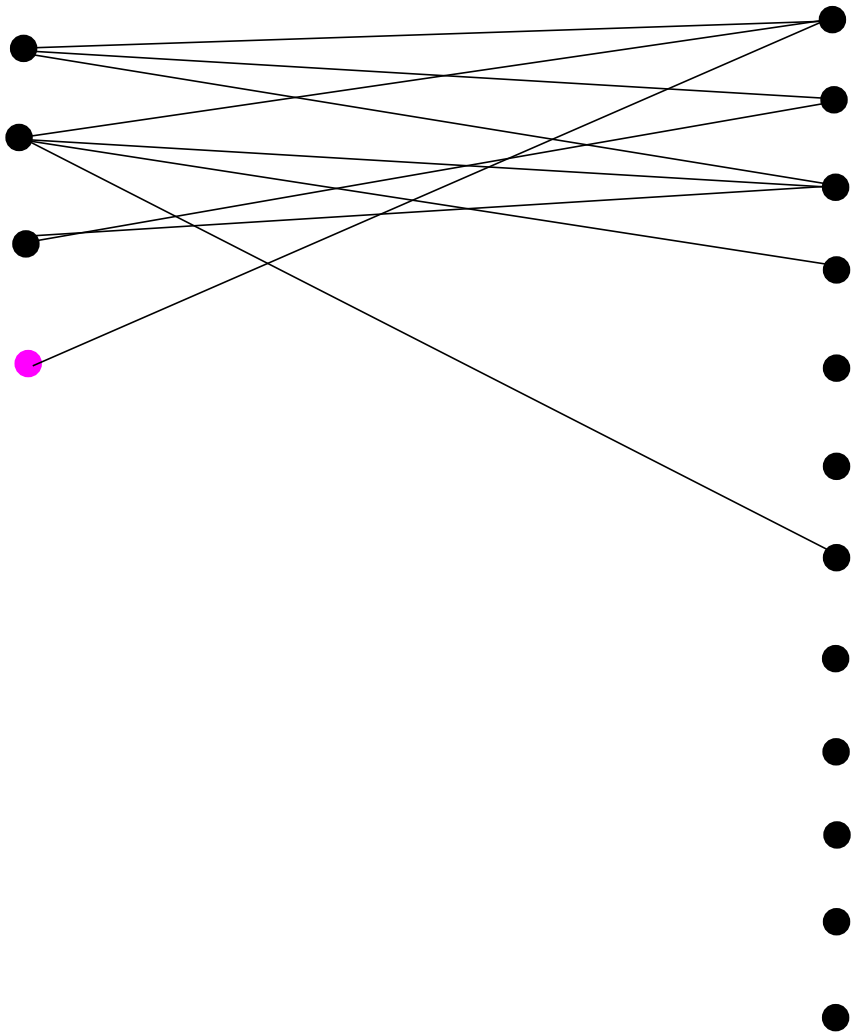

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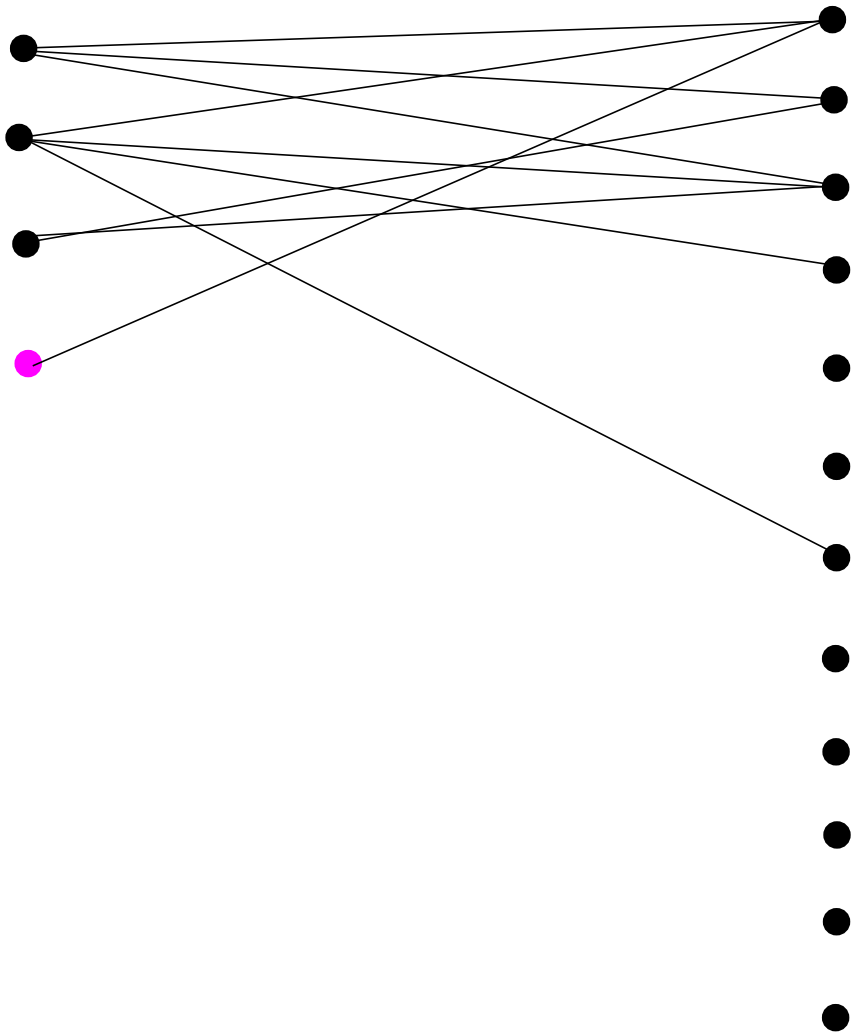

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## Main Analytic Question

Under what conditions does a newly inserted client $c$ have a short augmenting path in ALL possible matchings?

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Simplifying Assumption: It is always possible to match every client

## Server Necessities

Can ask about each server $s$: "how necessary is $s$ for ensuring that every client can be matched"

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## Example: Complete Bipartite Graph with $n$ servers and $n$ clients

All servers have necessity 1

## Example: Complete Bipartite Graph with $2n$ servers and $n$ clients

All servers have necessity $1/2$

# The necessity $\alpha(s)$ of a server $s$.

Client Server $\alpha(s)$



**Define a *flow* from each client to its neighbouring servers**

- Each client has out-flow 1
- the flow is *balanced*

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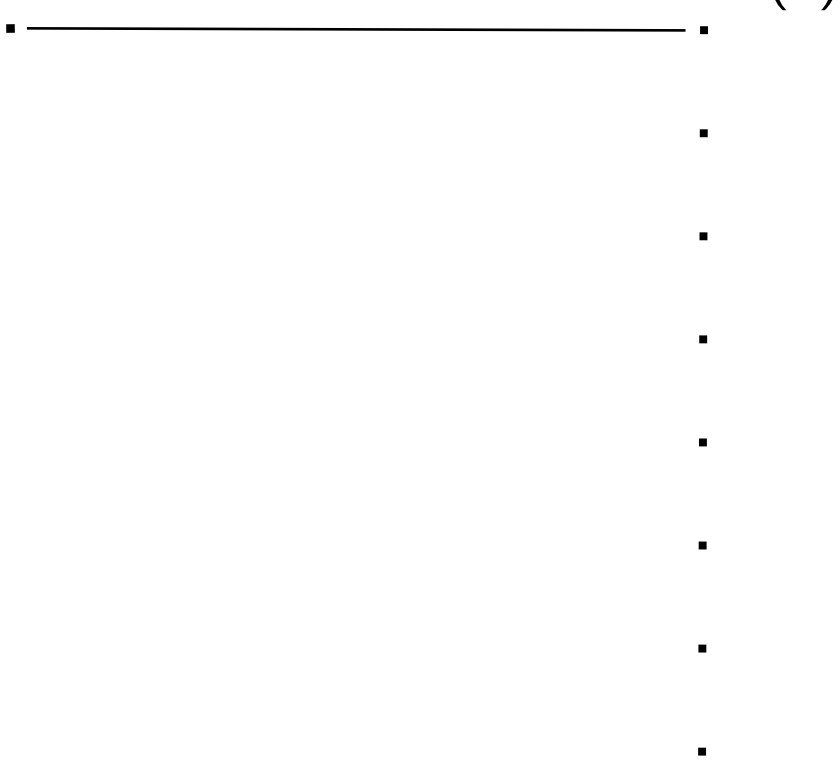


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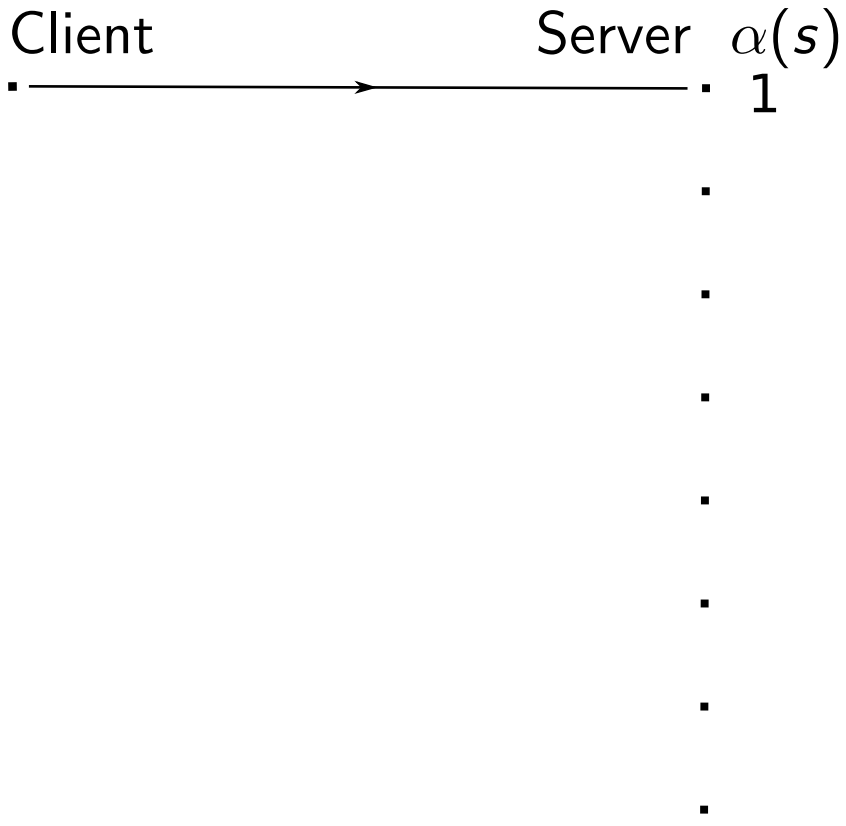


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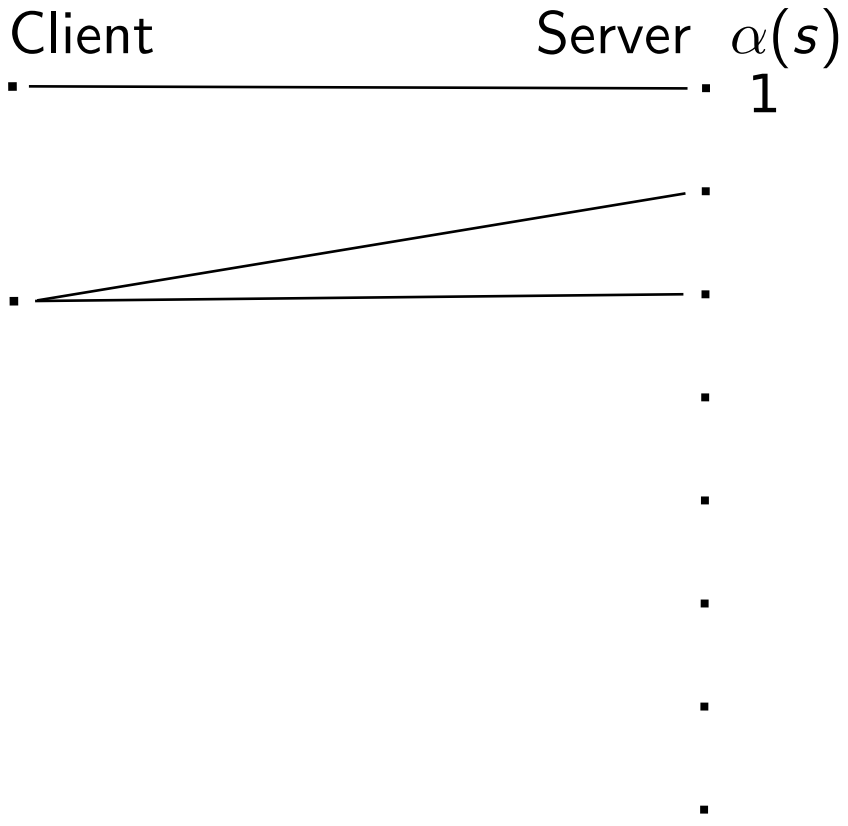


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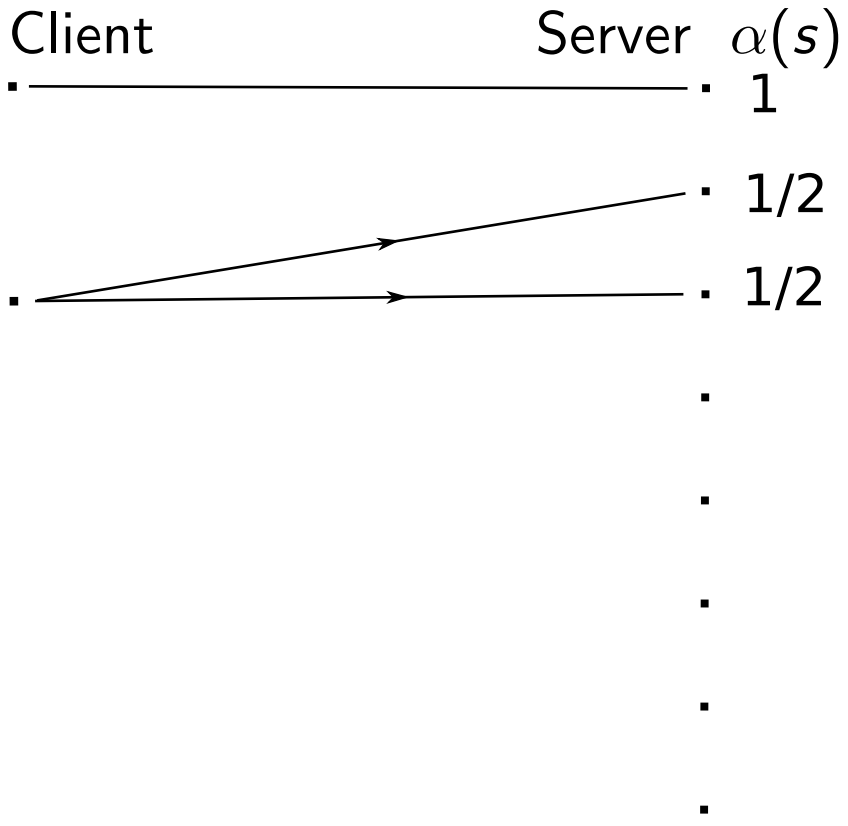


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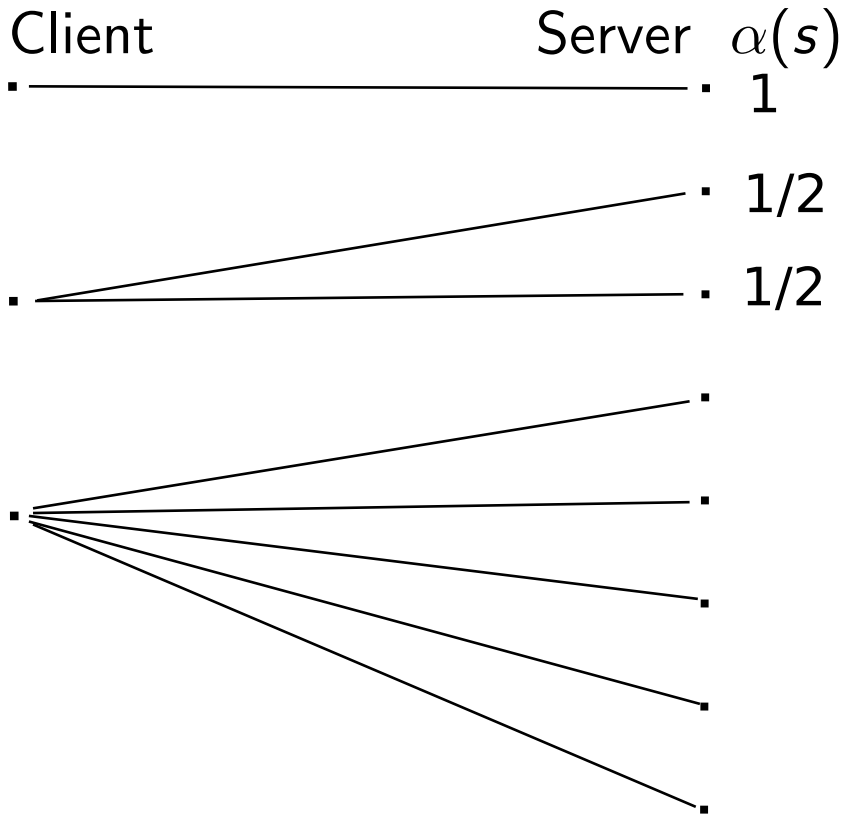


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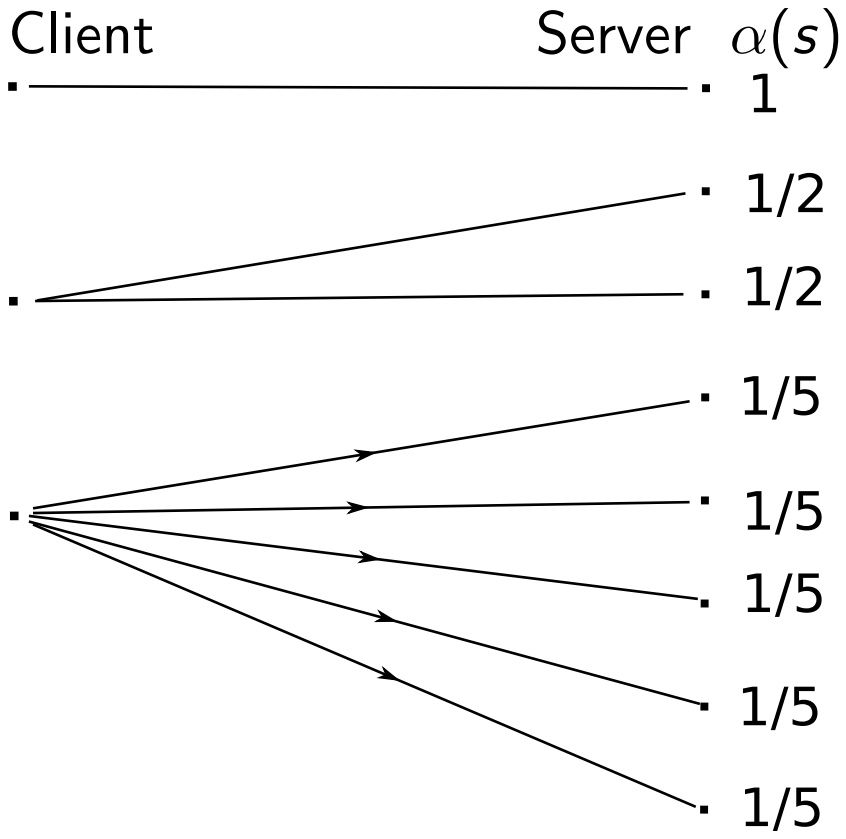


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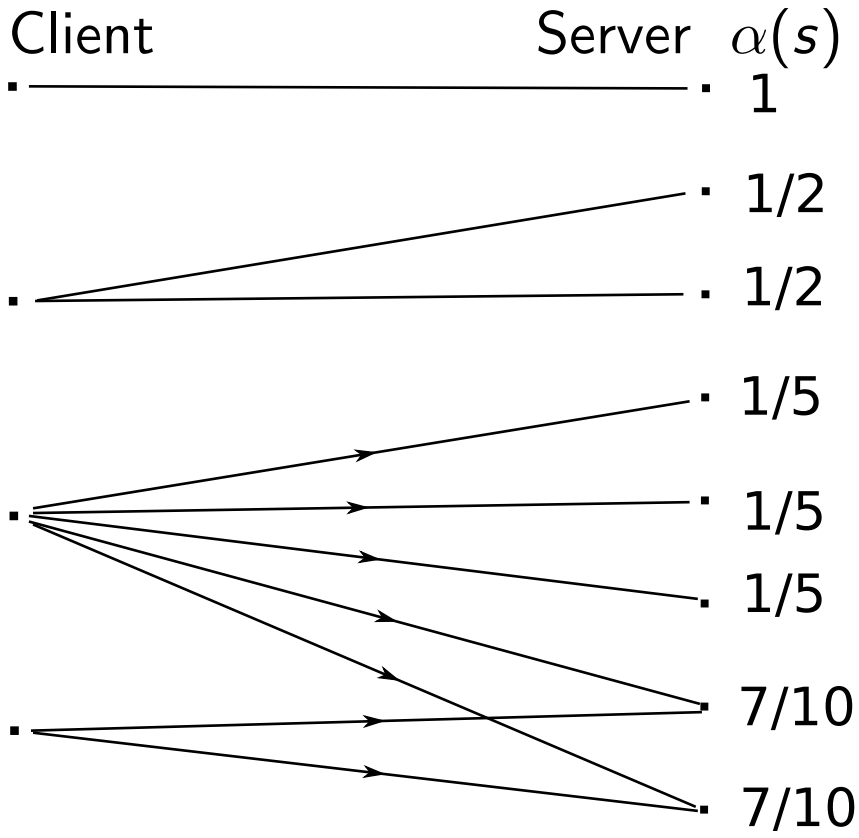


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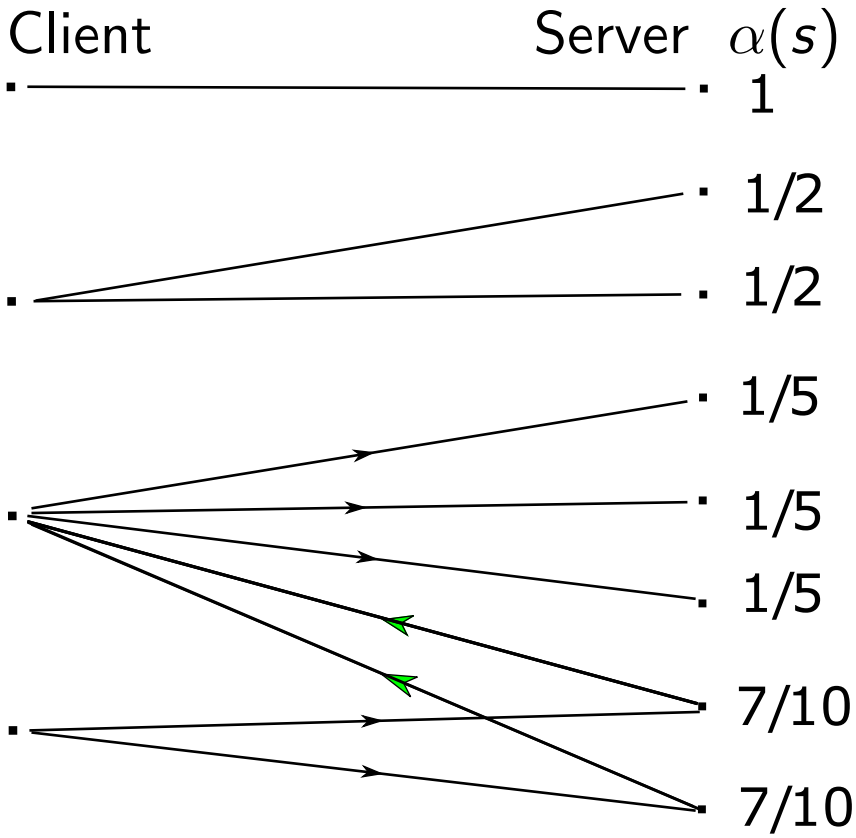


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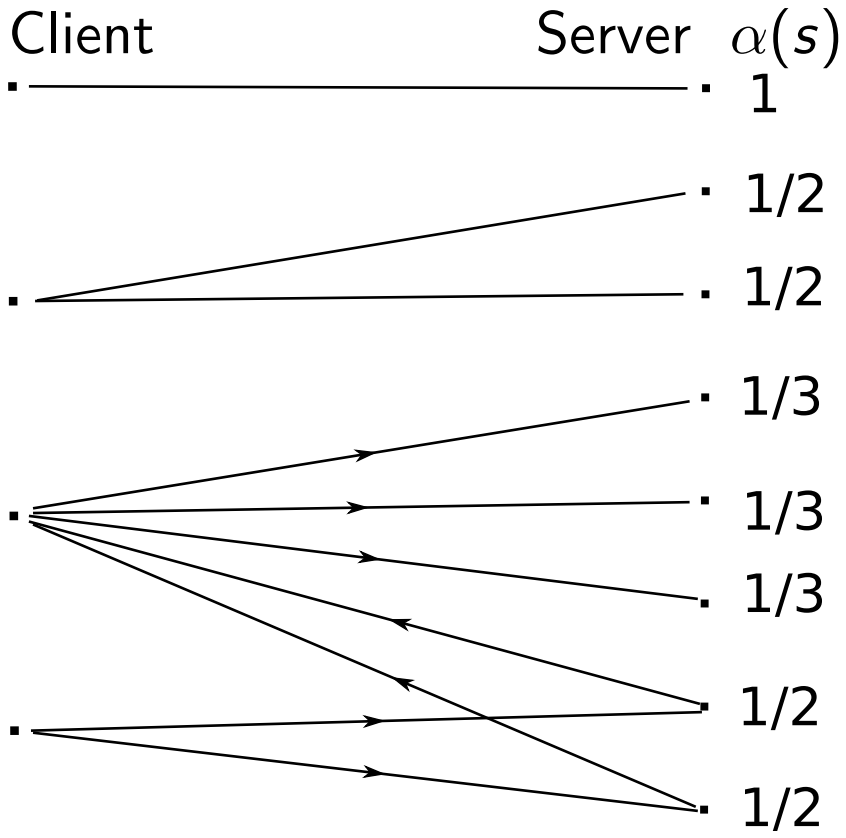


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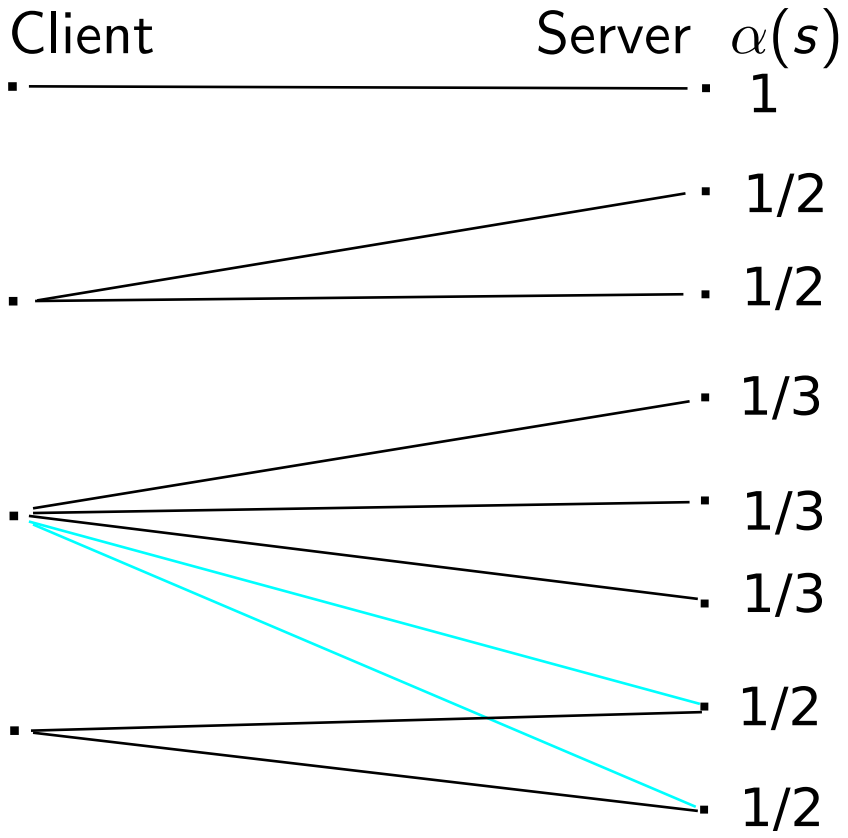


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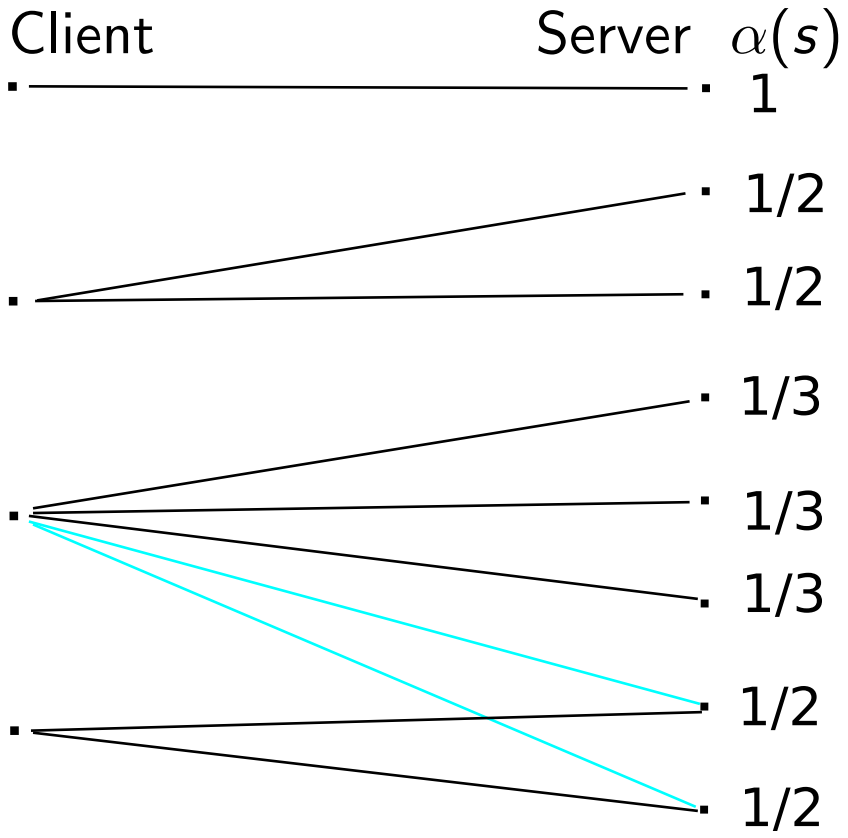


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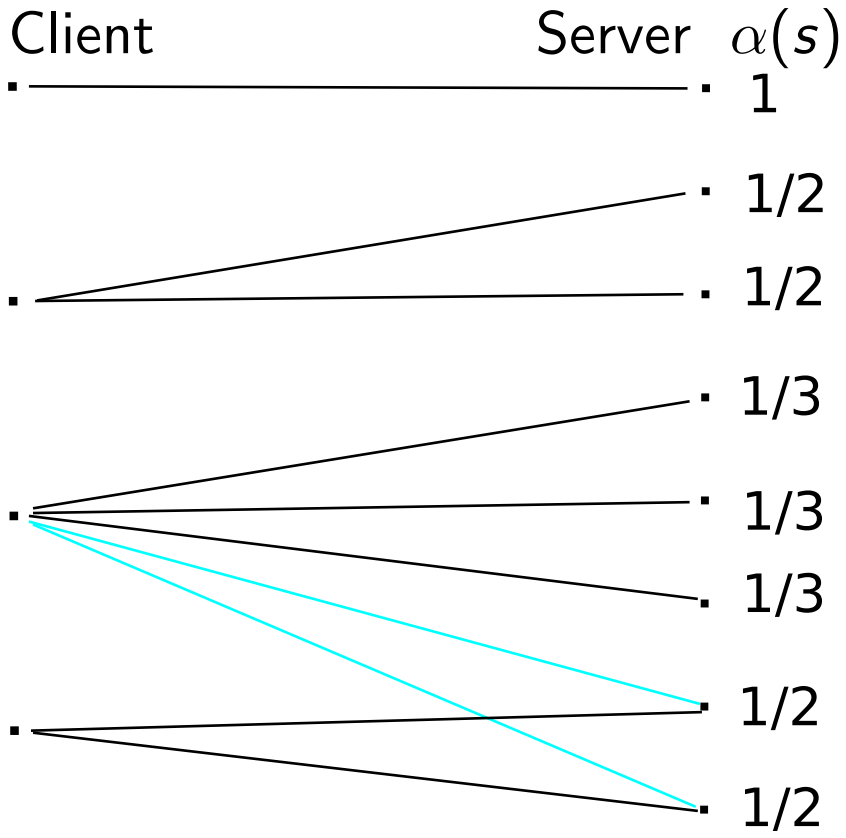


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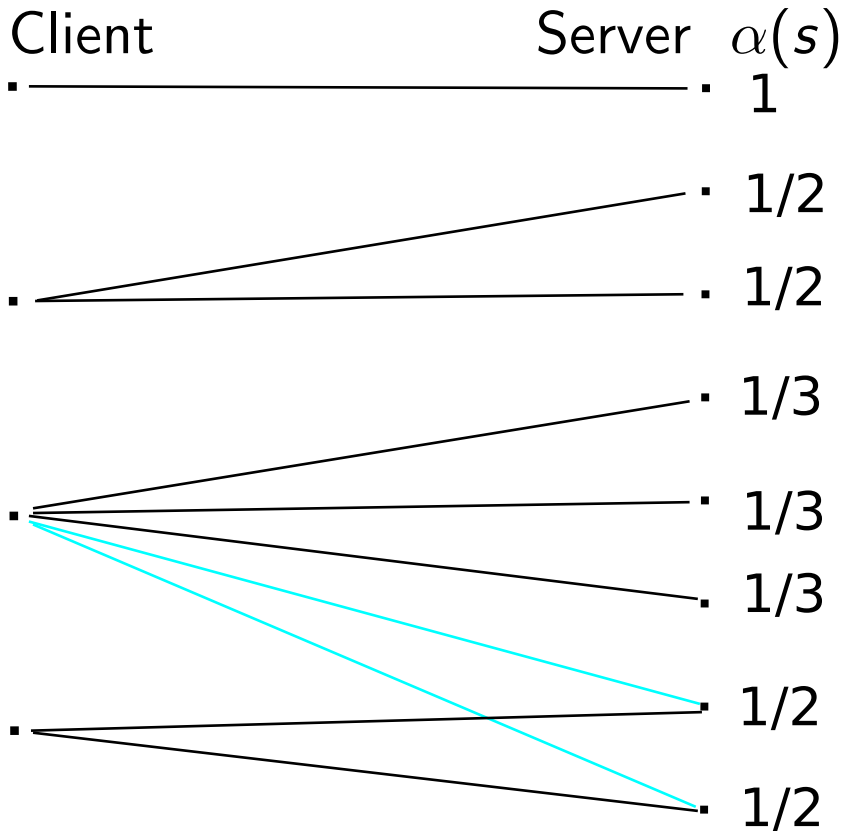


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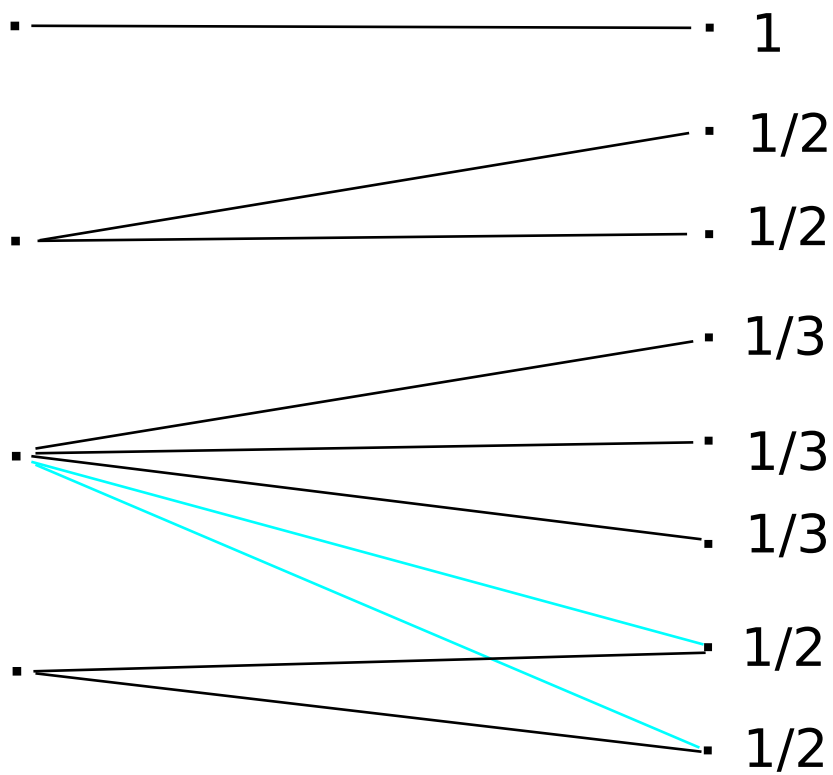
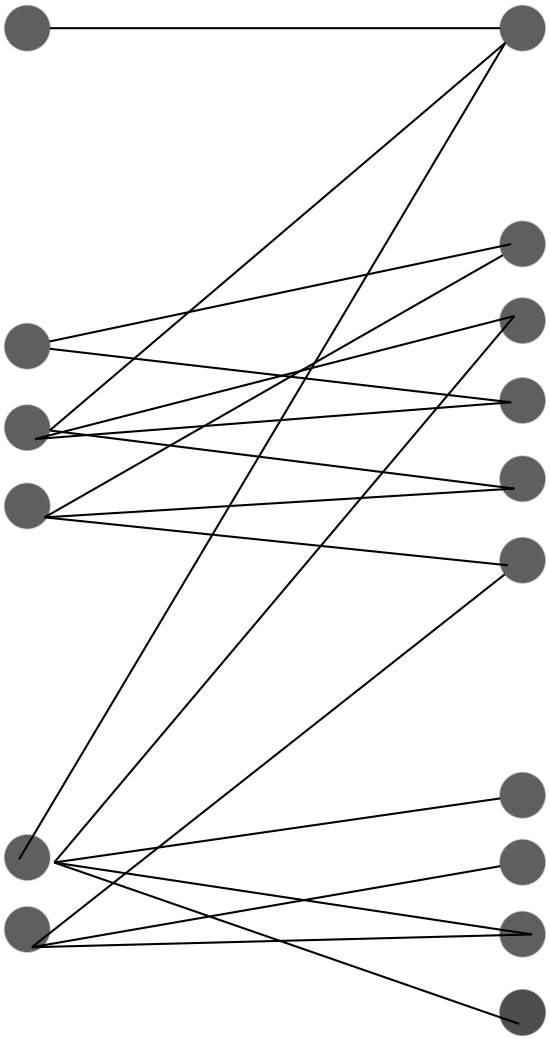
Observation: the necessities are independent of the current matching.

# Uniqueness $\rightarrow$ Structure?




Similar to graph decomposition from Goel, Kapralov, Khanna SODA 2012.

# Unique Server Necessities: combinatorial proof sketch



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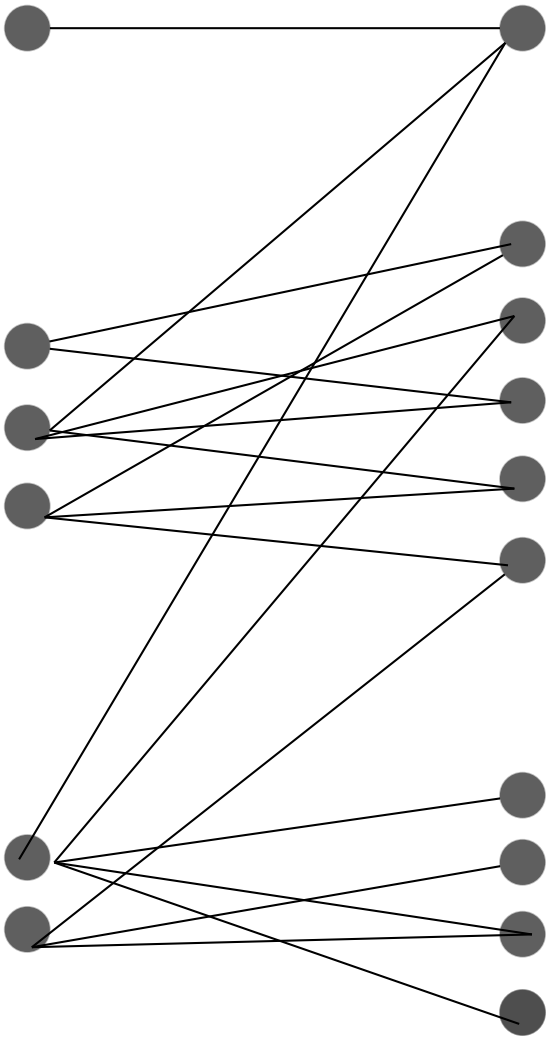

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All clients can be matched if every subset $K \subseteq C$ has expansion at least 1

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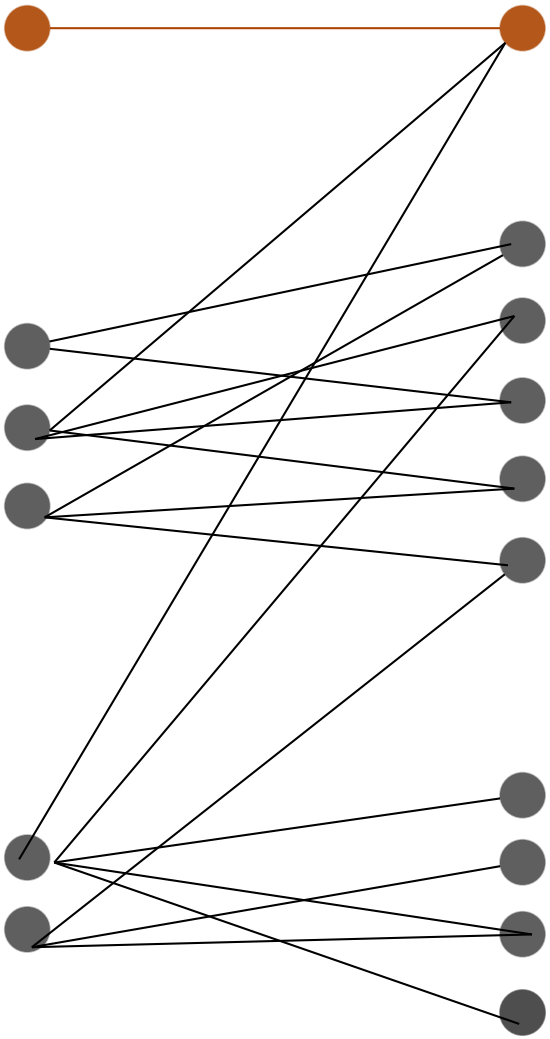

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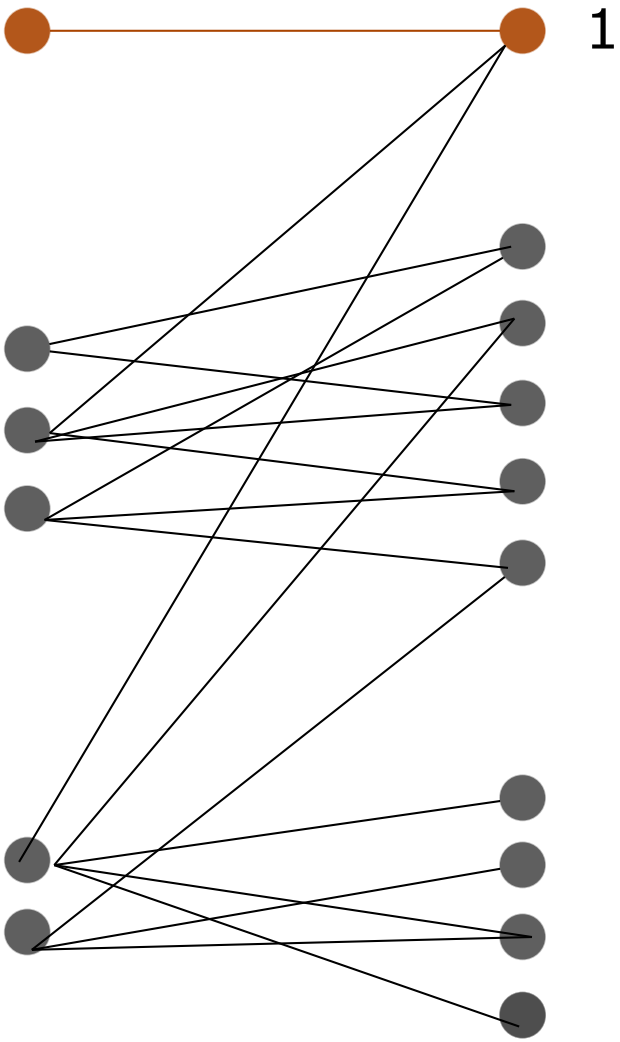


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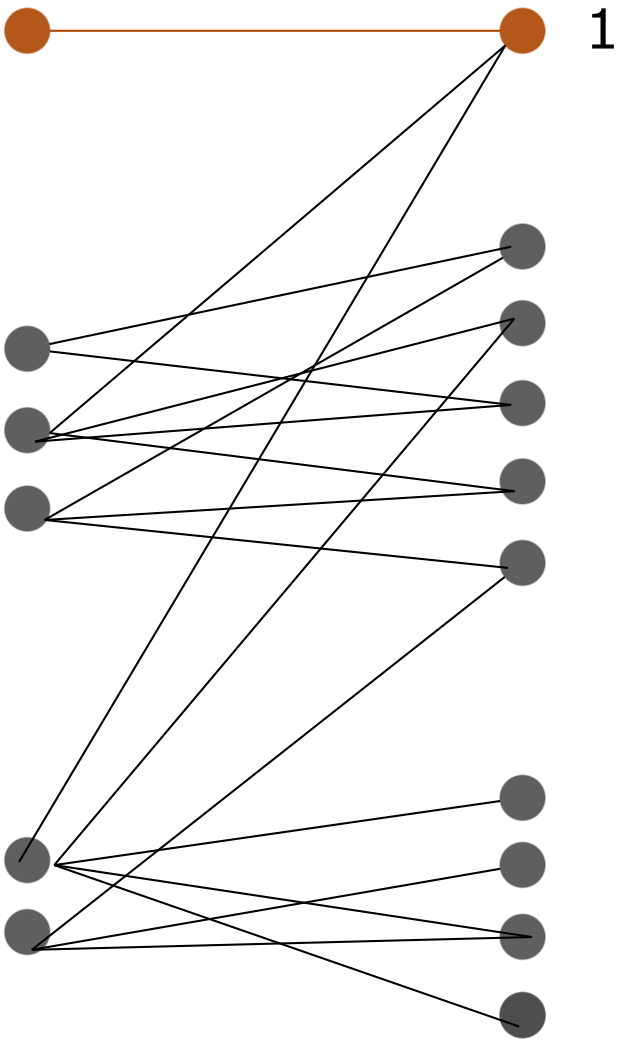


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## Our Decomposition

Successively Remove least expansive subset

- i.e. subset $K \subseteq C$ that minimizes $\frac{|N(K)|}{|K|}$.
- More expansion $\rightarrow$ lower necessity

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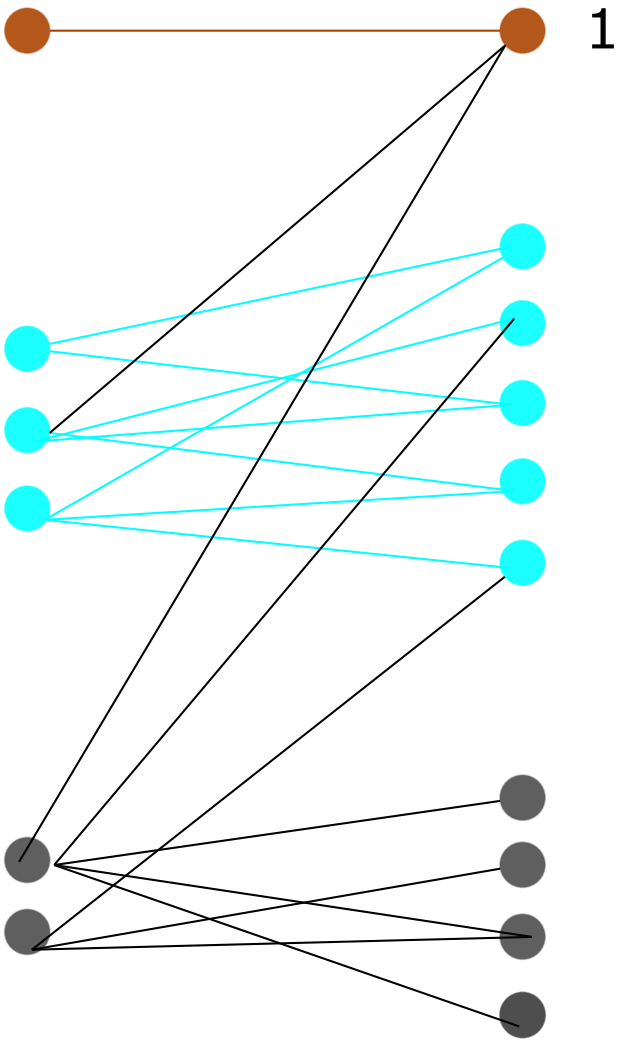


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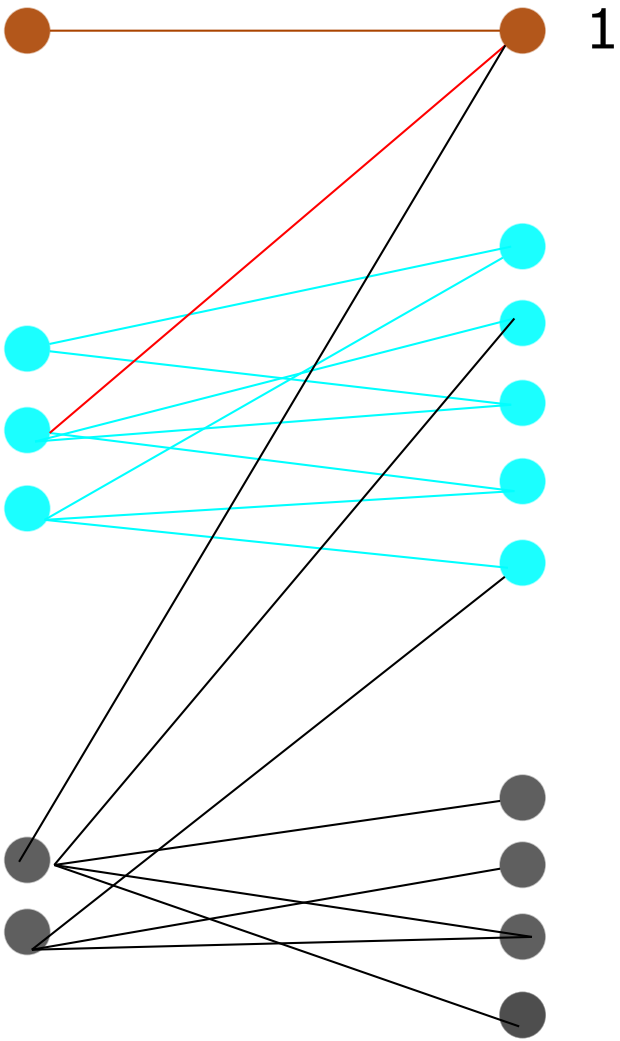


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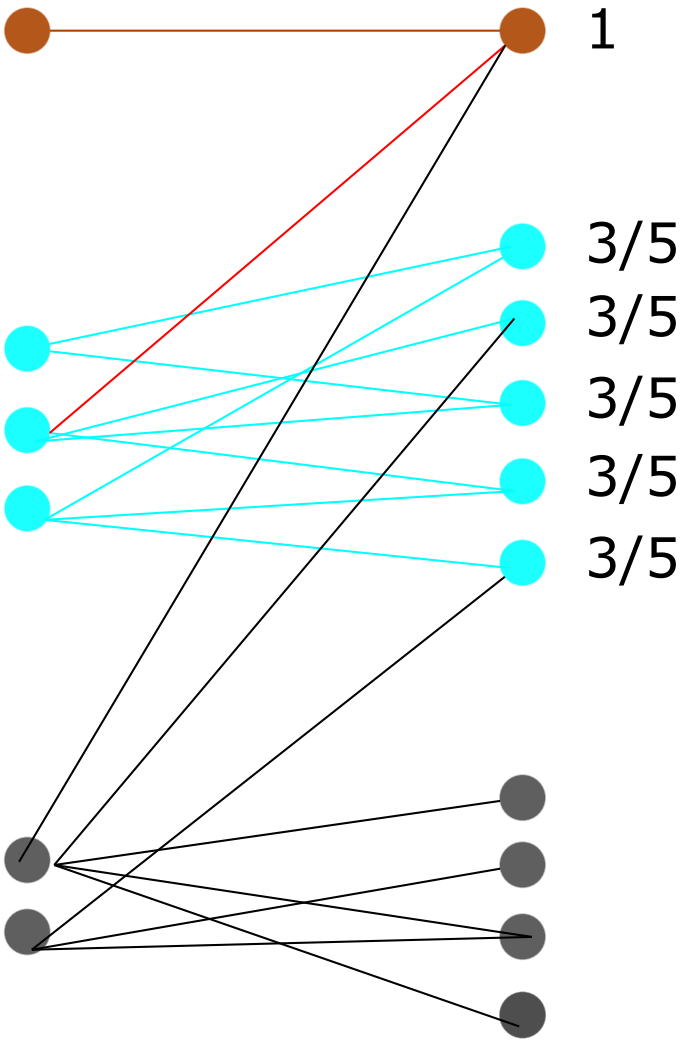


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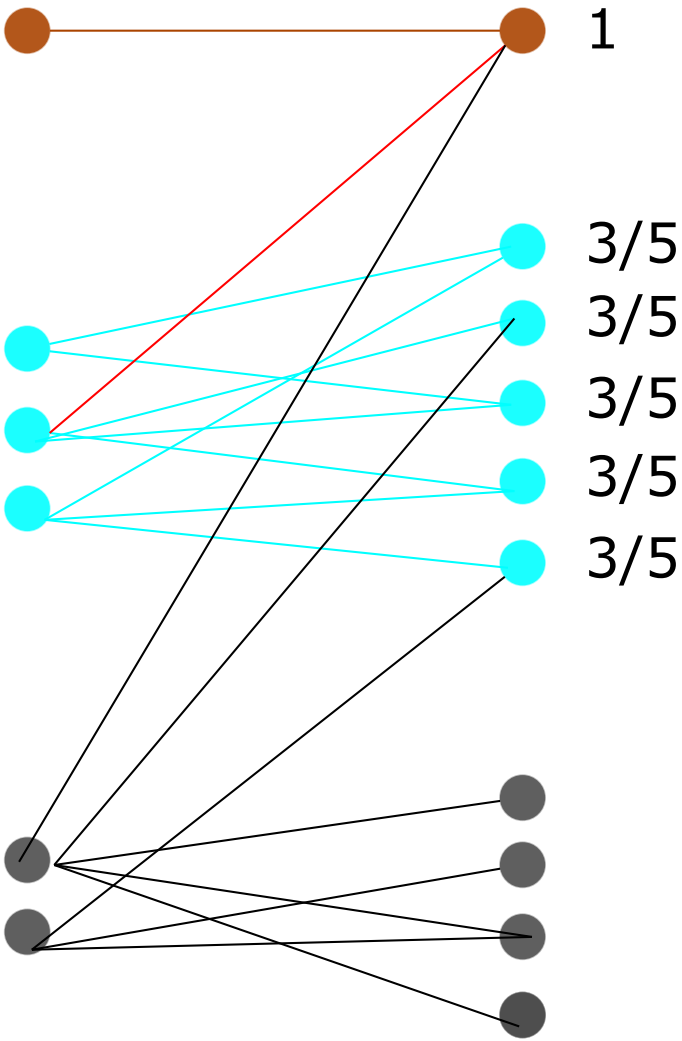


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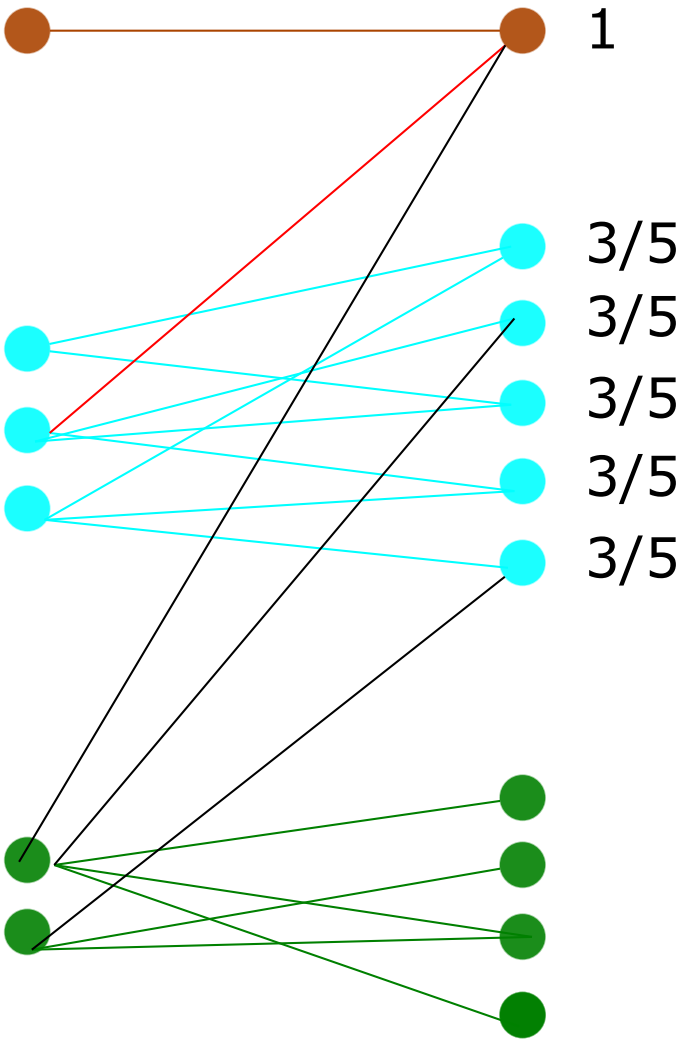


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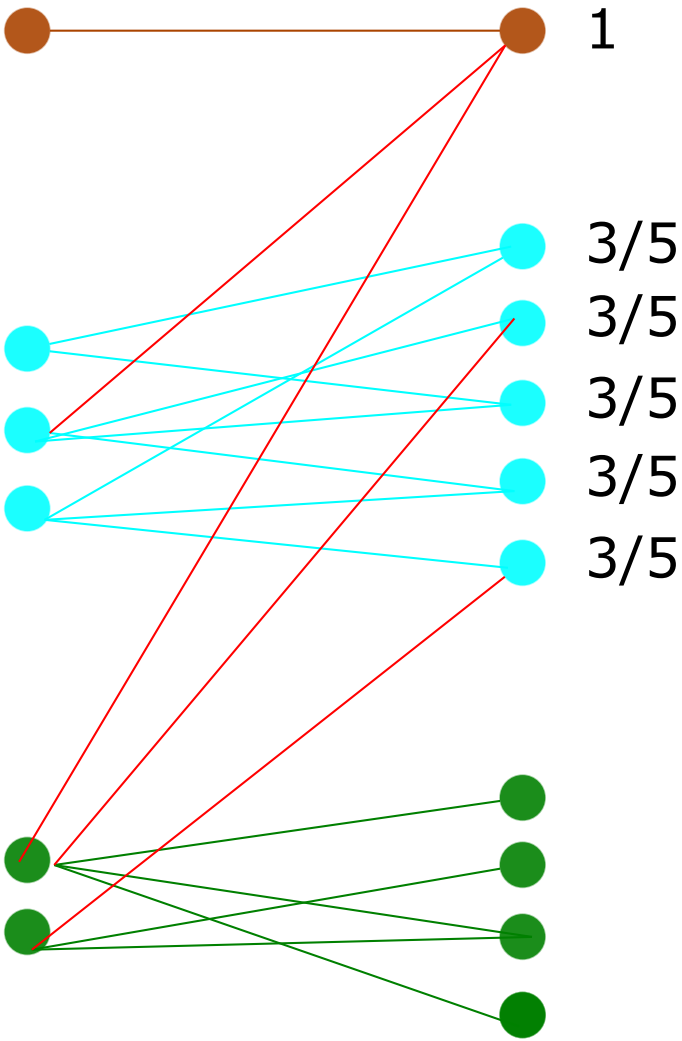


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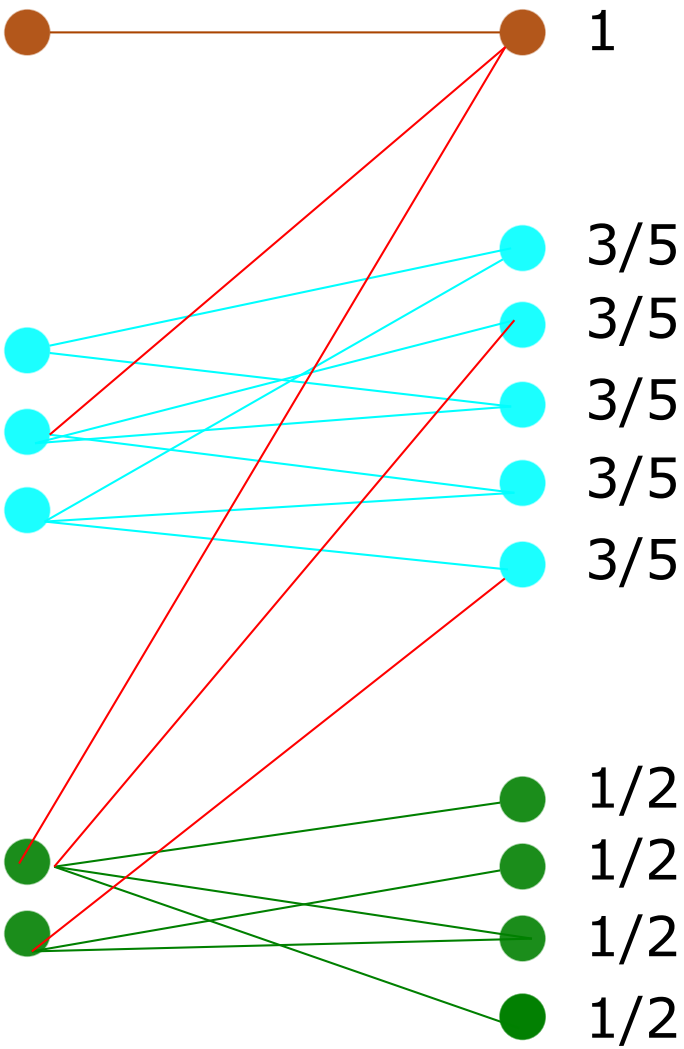


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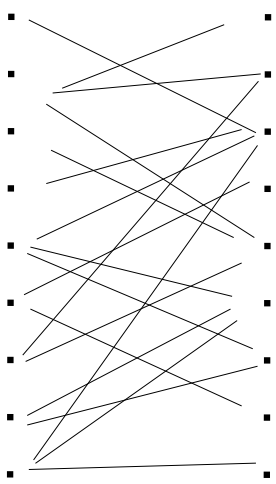
# Necessities and Hall's Theorem



## Theorem (Hall's Theorem)

*Can match all clients iff for every $K \subseteq C$, the neighbourhood $N(K)$ has size $|N(K)| \geq |K|$. In other words, $\frac{|K|}{|N(K)|} \leq 1$.*

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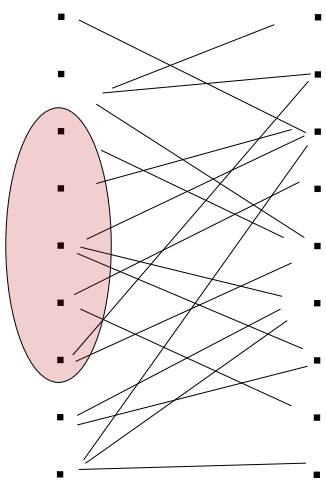

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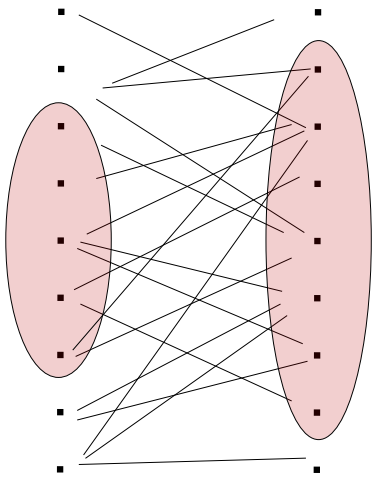

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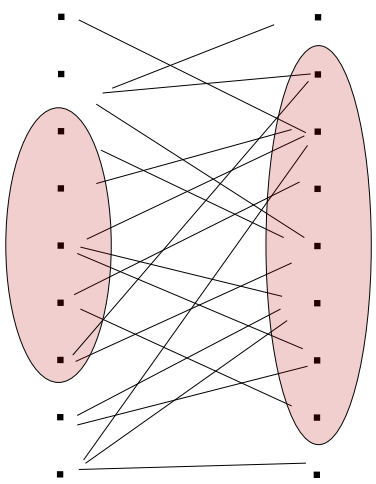

$\alpha(s) \leq 1$

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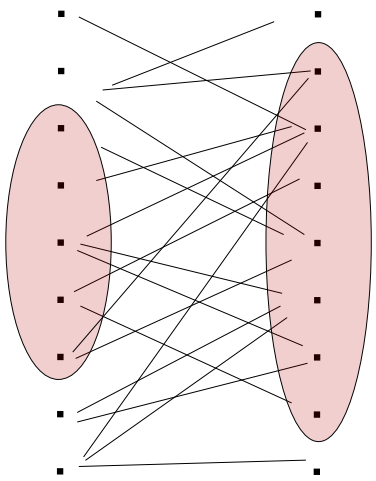

$$\alpha(s) \leq 3/4 \implies \frac{|K|}{|N(K)|} \leq 3/4$$

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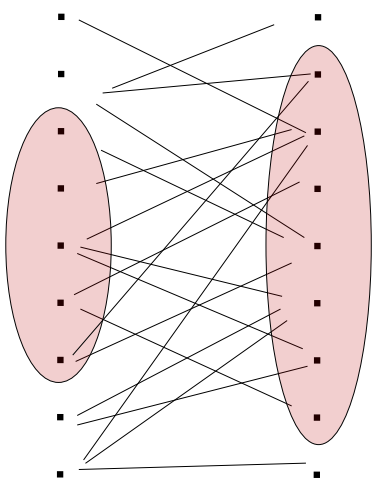

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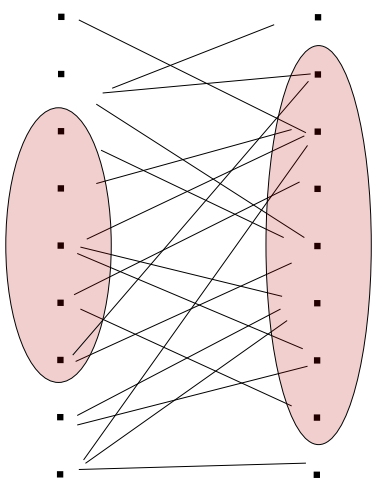

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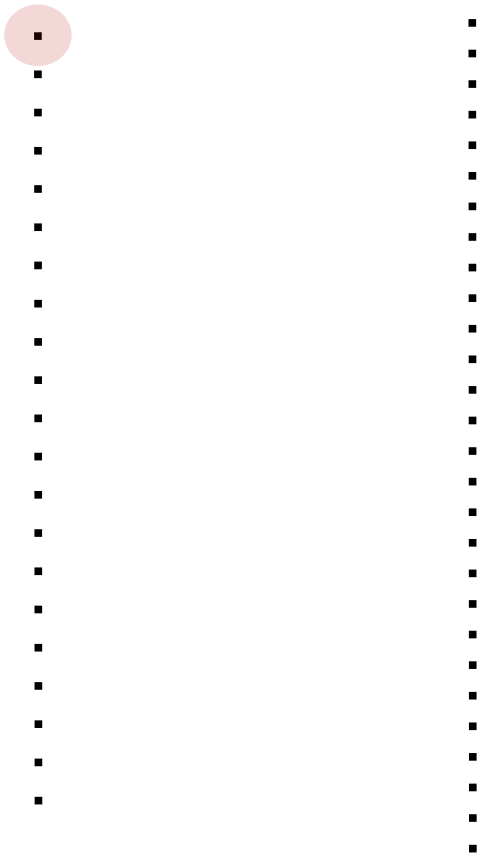
## Lemma

*Say that for $K \subset C$, $\alpha(s) \leq \alpha_K^{max}$ for all $s \in N(K)$. Then $\frac{|K|}{|N(K)|} \leq \alpha_K^{max}$*

- *Expansion Factor:* $1/\alpha_K^{max}$

# Outline

# Expansion Lemma



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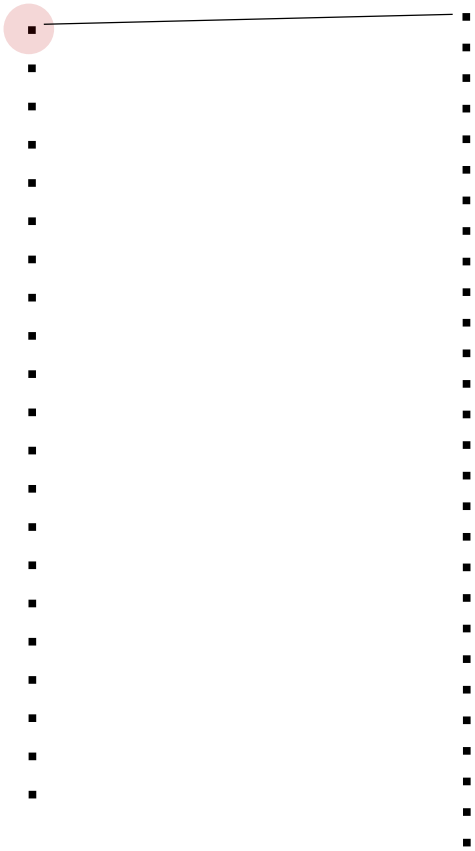

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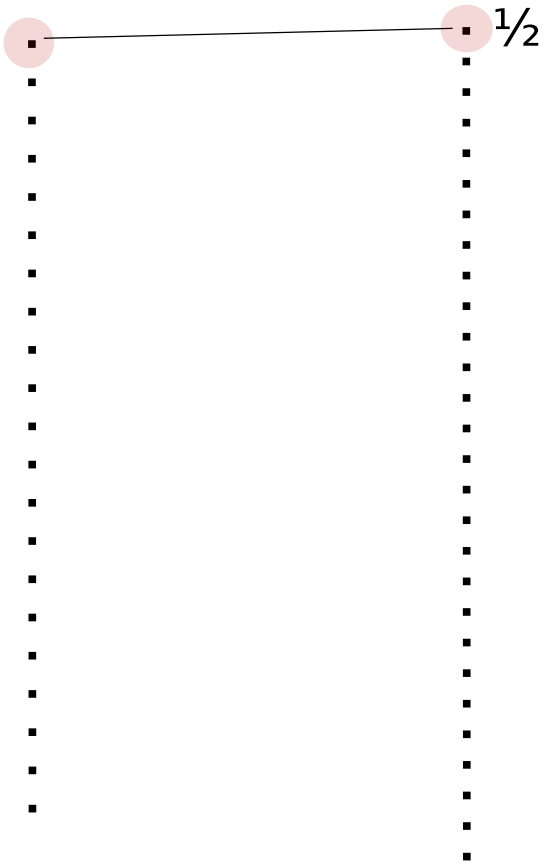

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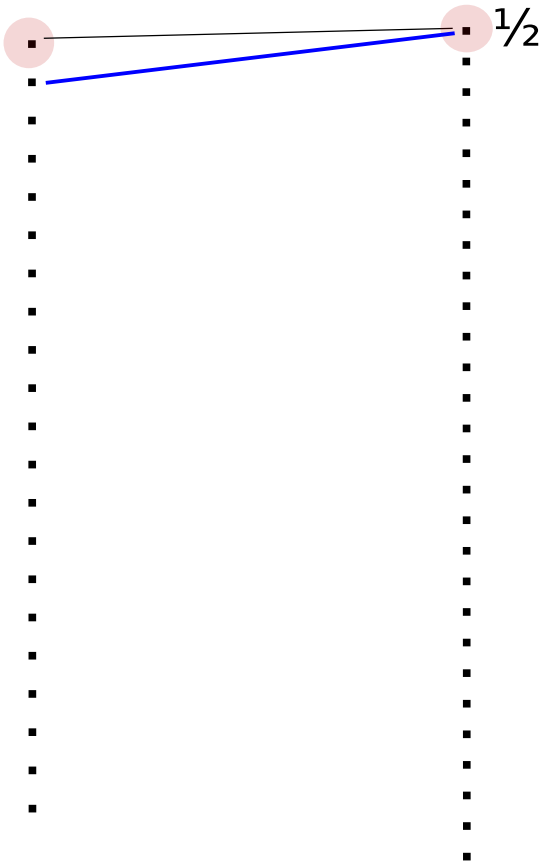

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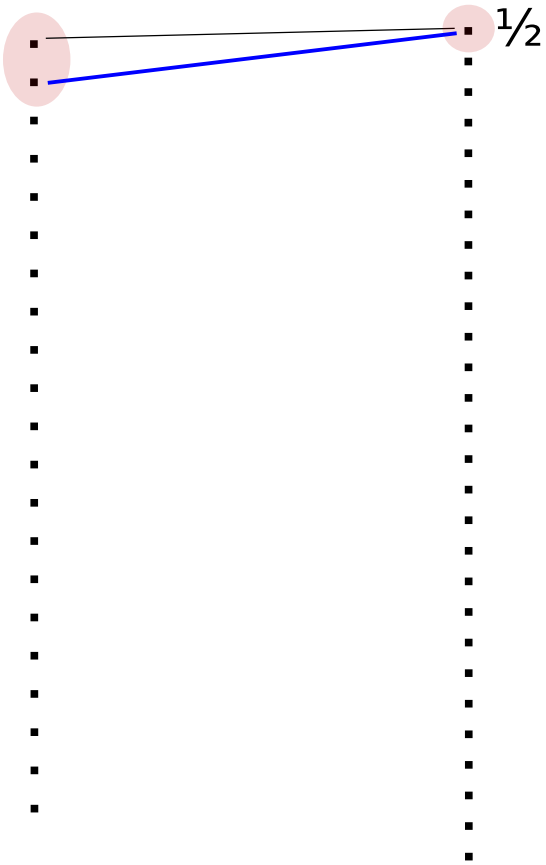

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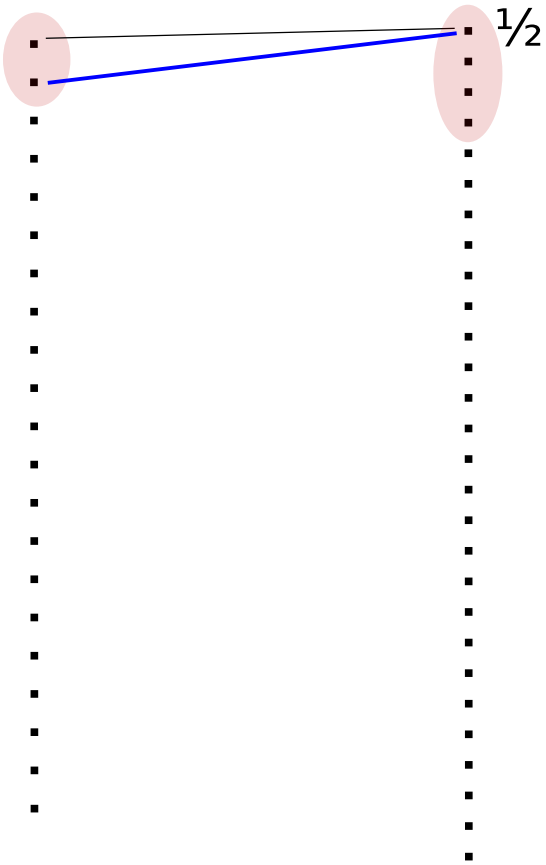


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- **Expansion Factor:** $\frac{1}{\alpha^{max}} = 2$
- **Balanced Flow:** $\alpha^{max}$ never increases as we search for the augmenting path

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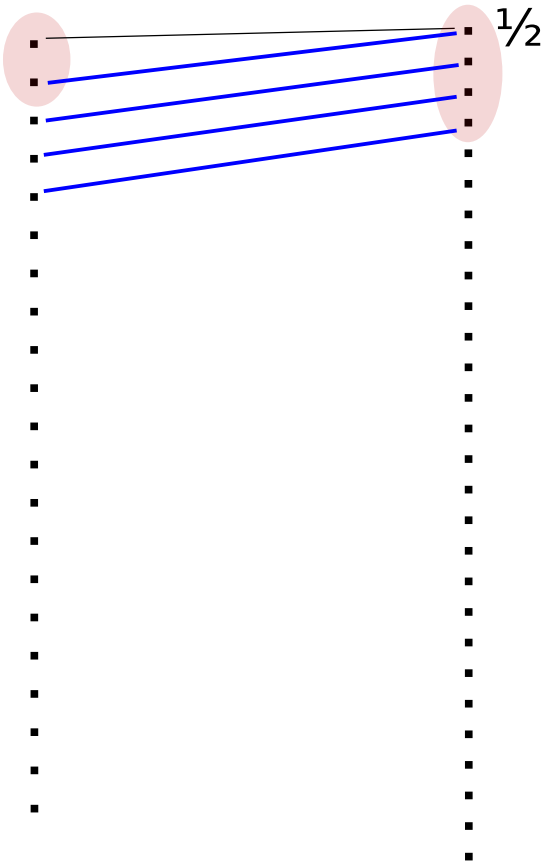


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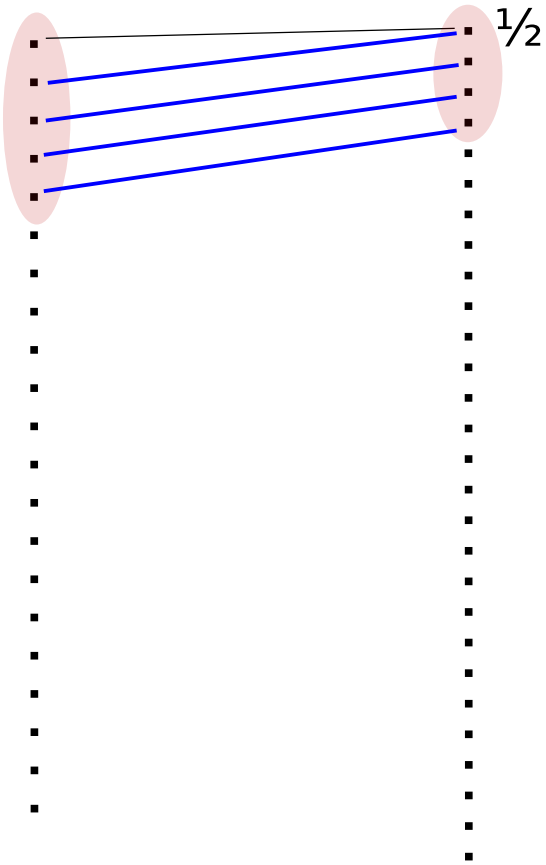


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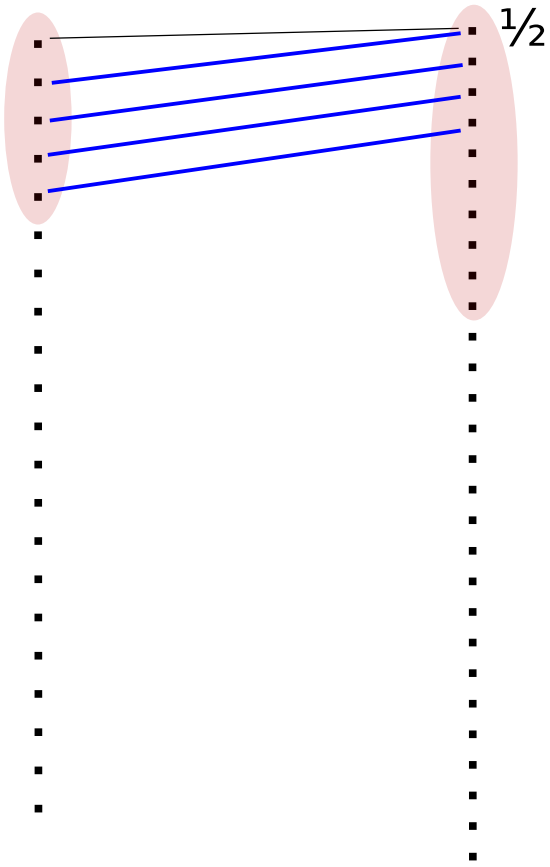


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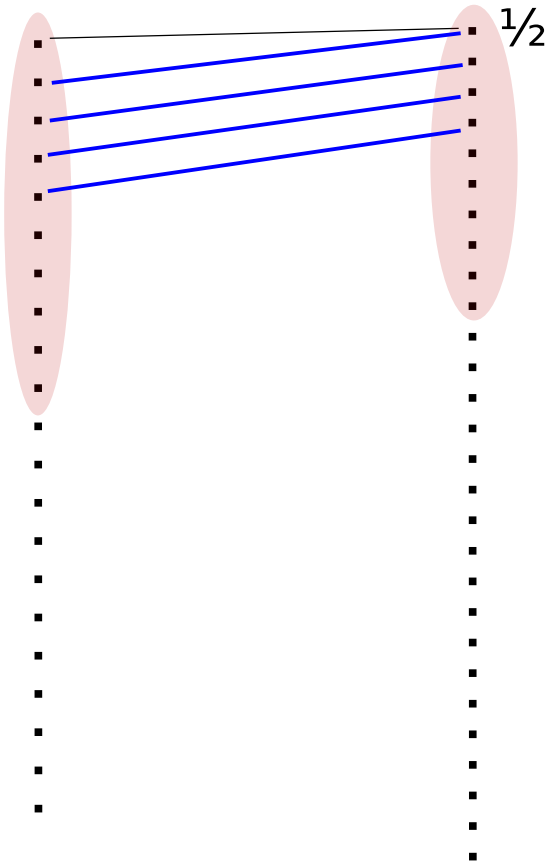


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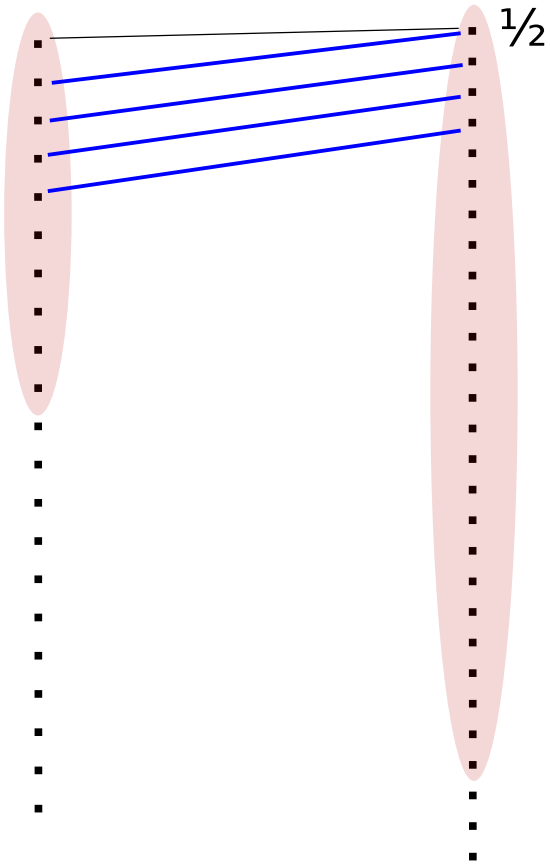


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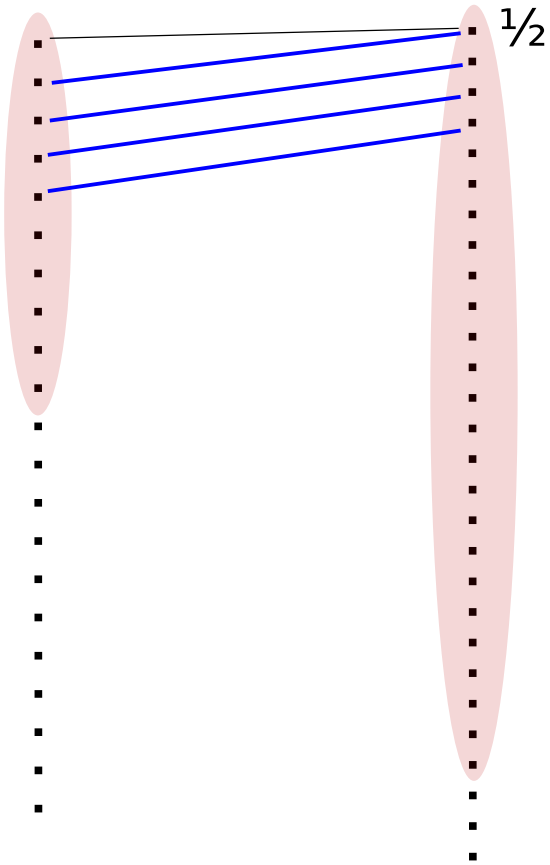


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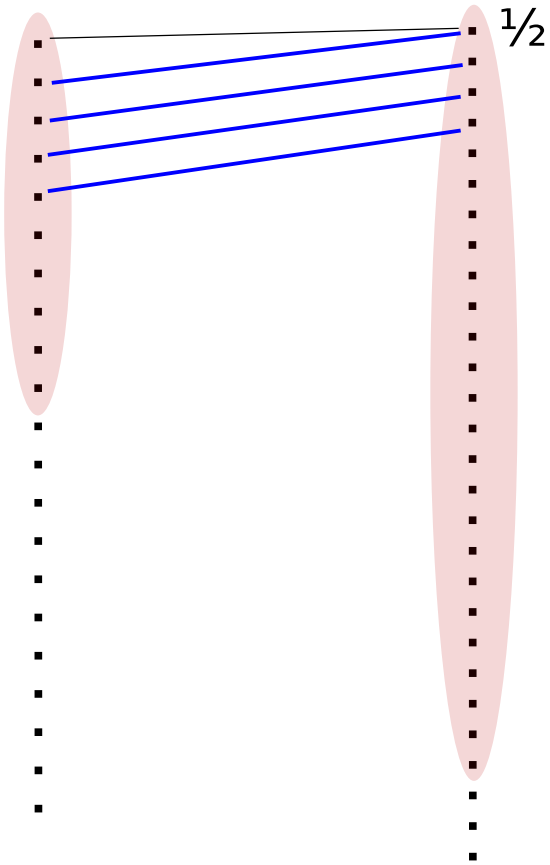


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## Proof.

Expansion factor of $\frac{1}{1-\epsilon} \sim 1 + \epsilon \Longrightarrow$
path has length $O(\log_{1+\epsilon} n) = O(\frac{\log n}{\epsilon})$ □

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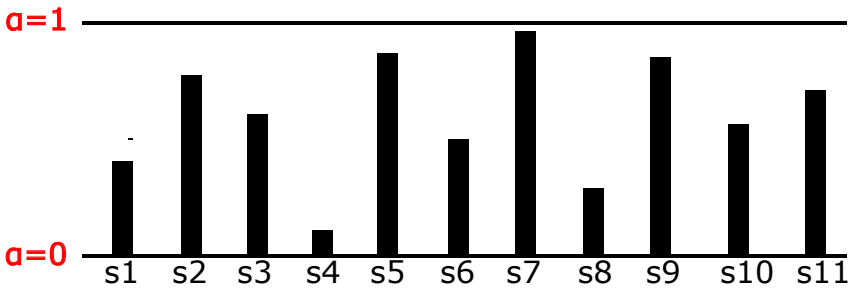

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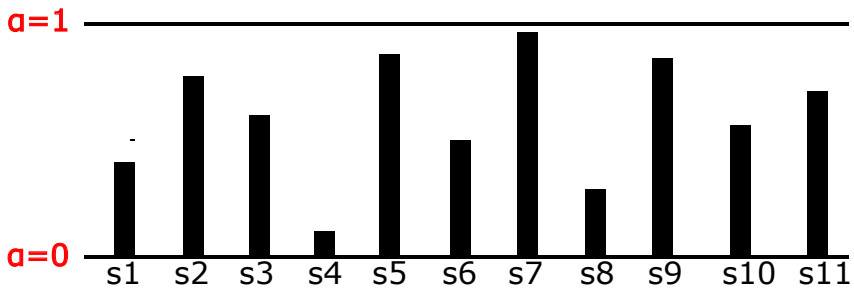


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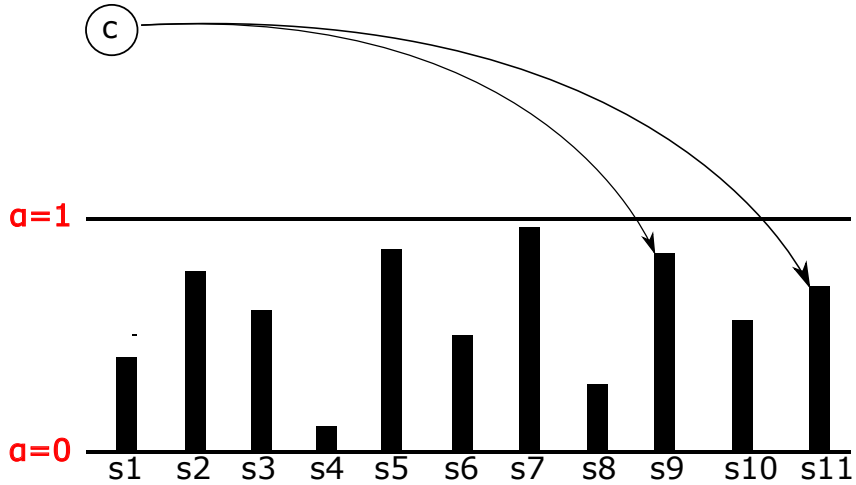


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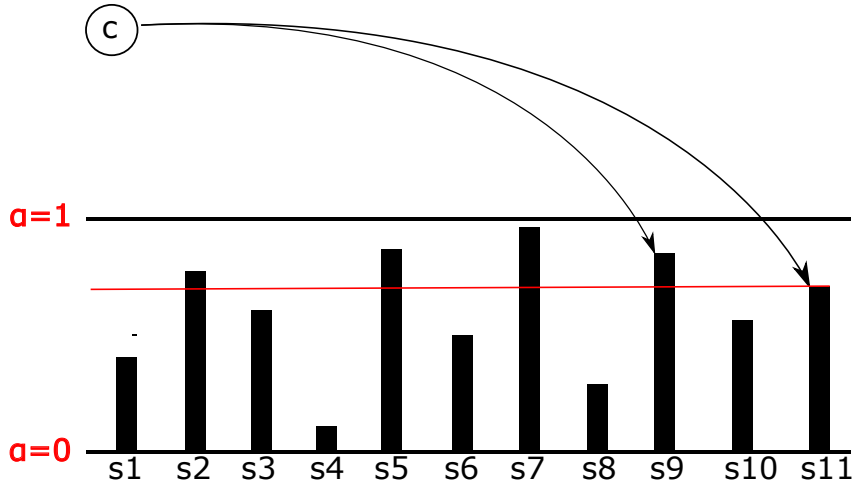


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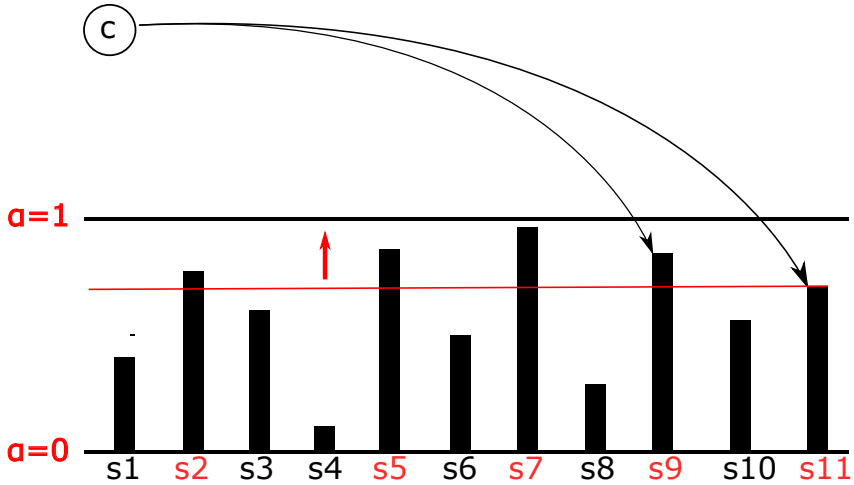


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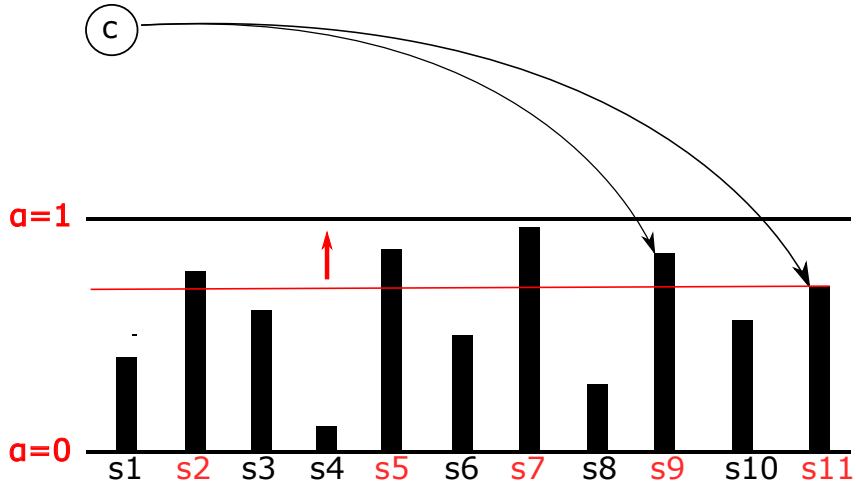


## $\alpha(s)$ changes over time

- each client adds 1 to $\sum \alpha(s)$
- $\alpha(s)$ non-decreasing
- Flow Only Goes Upwards

## Expansion Lemma

$\alpha(s) = 1 - \epsilon \rightarrow$ path length $O(\frac{\log n}{\epsilon})$

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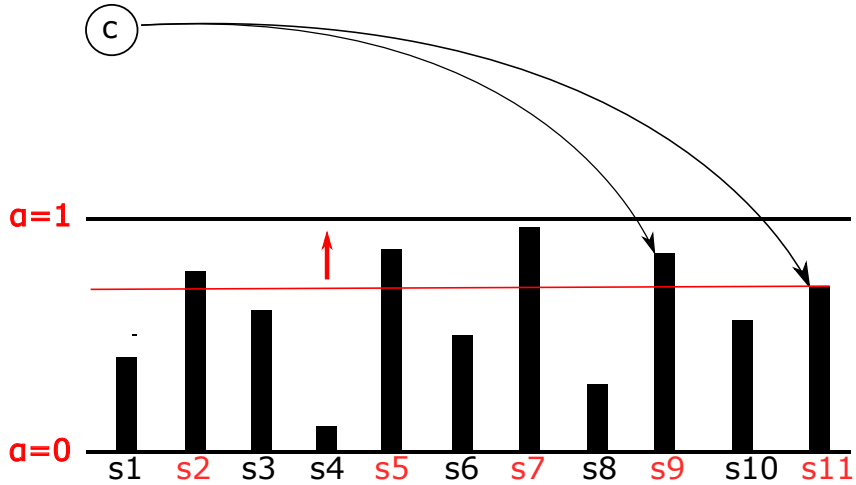


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## Example: $\alpha(s11) \sim \frac{9}{10}$

- Expansion Lemma: path length $= O(\log n/10) = O(\log(n))$
- at most $n$ augmenting paths in total (trivial)
- total number of changes: $O(n \log n)$

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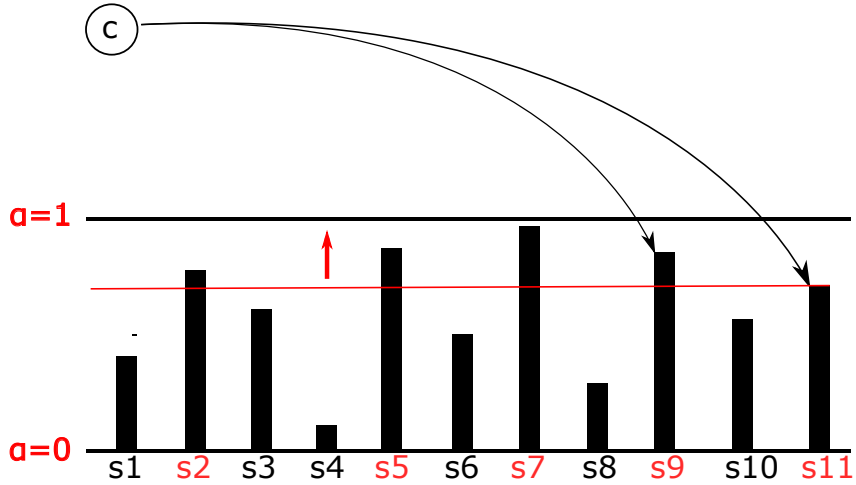


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## Example: $\alpha(s11) \sim 1 - \frac{1}{n^{1/3}}$

- augmenting path length $= O(n^{1/3} \log n)$
- all new flow goes to servers with $\alpha(s) \geq 1 - \frac{1}{n^{1/3}}$

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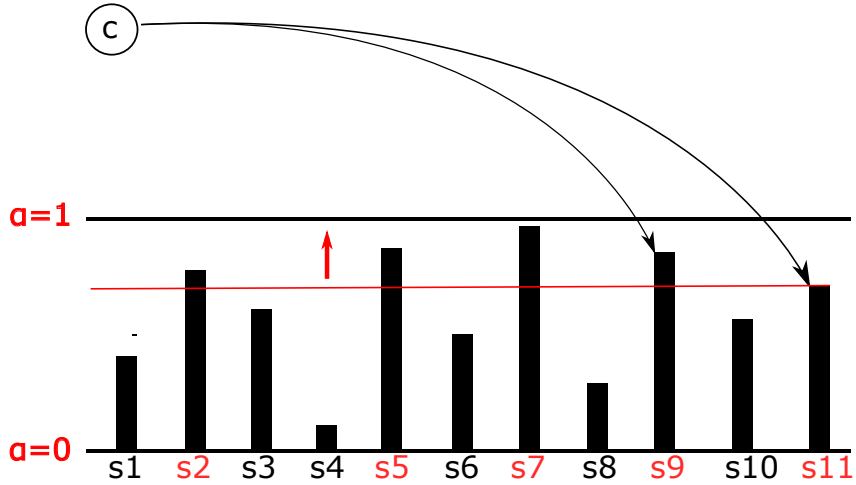


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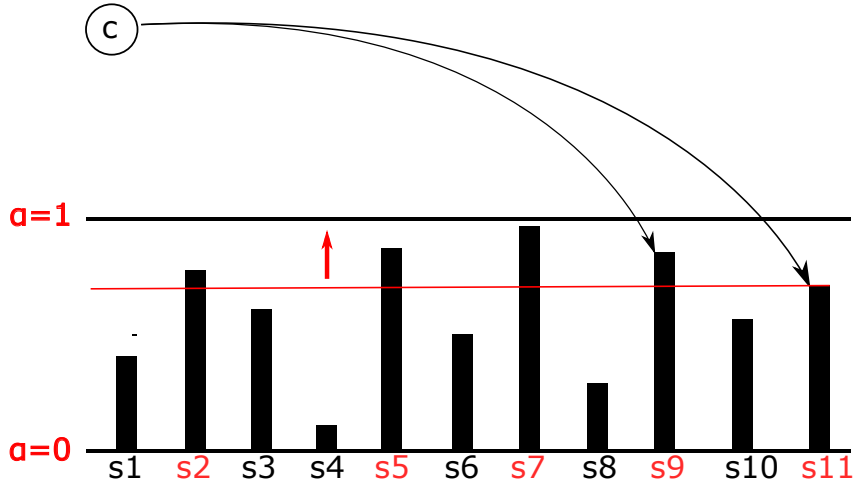


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- at most $n^{2/3}$ such paths $\rightarrow (n^{2/3})(n^{1/3} \log n) = O(n \log(n))$ changes

# Outline

Lower Bound: $\Omega(n \log n)$ total number of server changes (amortized $\log n$).

## Theorem (number of changes)

*SAP makes at most $O(n \log^2 n)$ replacements in total (amortized $\log^2 n$).*

- *Previous: SAP makes $O(n^2)$ replacements (trivial)*
- *Previous: SAP makes $O(n \log n)$ replacements in special cases*
- *Previous: non-SAP protocol with $O(n^{1.5})$ replacements*

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## Theorem: Minimizing Max Server Load

- Let $L$ be maximum server load in final graph
- Lower Bound: $\Omega(\min\{nL, n^{1.5}\})$ server changes
- Upper Bound: $O(\log^2 n \min\{nL, n^{1.5}\})$ server changes

# Open Problems

Lower Bound: $\Omega(n \log n)$ total number of server changes (amortized $\log n$).

## Conjecture

*SAP makes at most* $O(n \log n)$ *server changes in total.*

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*There is an implementation of SAP that runs in* $O(m\sqrt{n})$ *total time.*