Approximation Algorithms for ℓ_0-Low Rank Approximation

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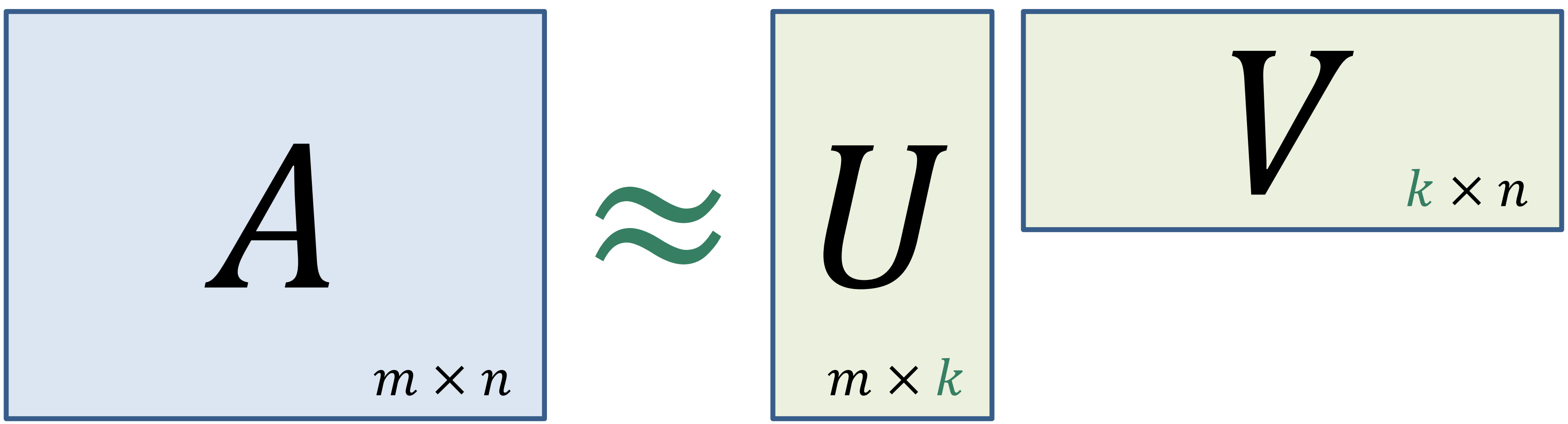
Low Rank Approximation



Motivation:

- Compact Storage
- Faster Matrix-Vector Multiplication
- Removes Outliers/Anomalies

(Compact Storage and Faster Matrix-Vector Multiplication): $mn \mapsto k(m+n)$

ℓ_0-Low Rank Approximation

$$\min_{\substack{U \in S^{m\times k} \\ V \in S^{k\times n}}} \left| A - U V \right|_{\mathbf{0}}$$

$A \in S^{m\times n}$, U: $m \times k$, V: $k \times n$

Input: semiring/field S, matrix $A \in S^{m\times n}$, positive integer k

Output: rank-k matrix decomposition U, V *minimizing* the
# of disagreeing components -- $|M|_0 = nnz(M)$

ℓ_0-Low Rank Approximation

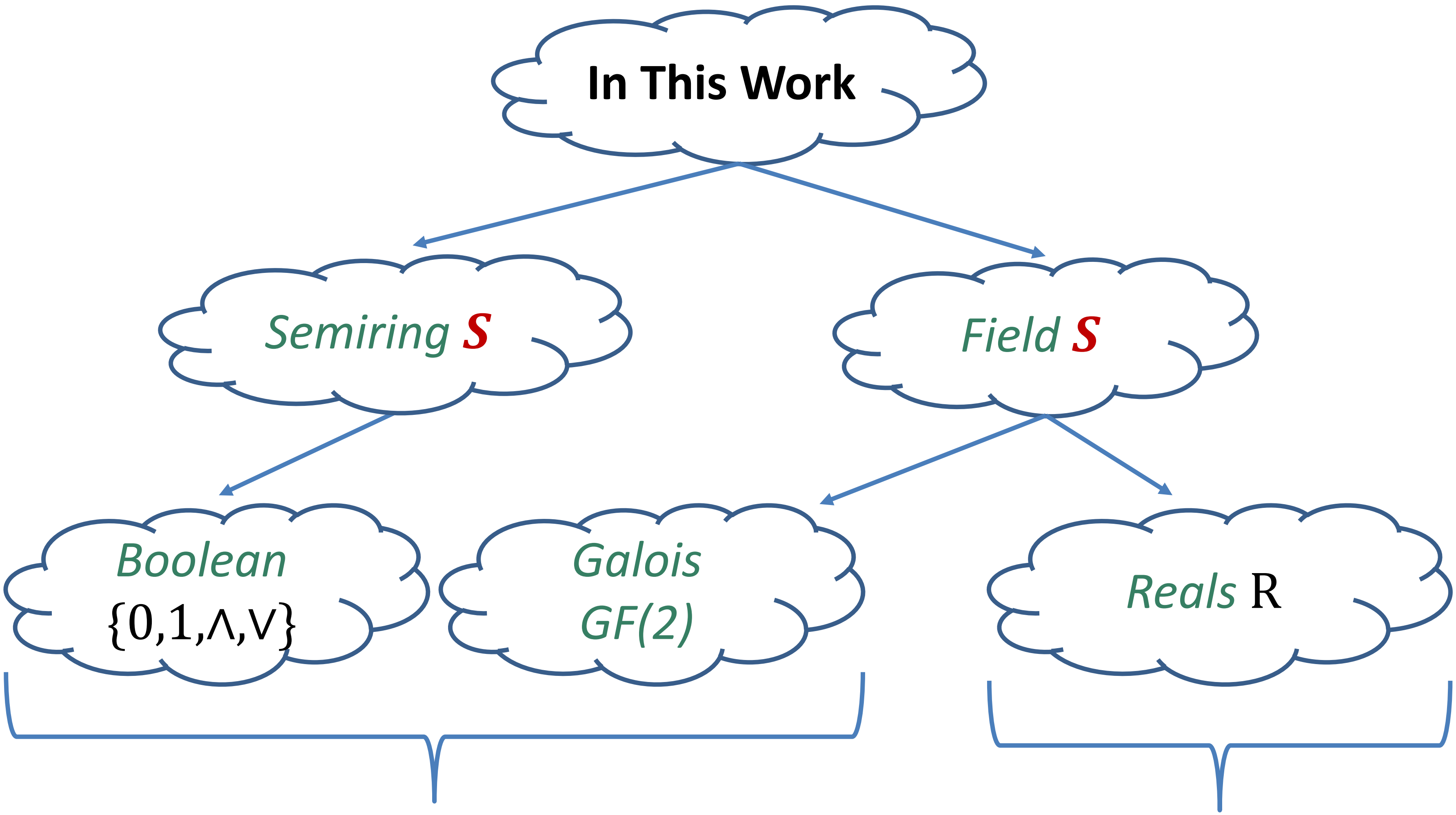
$$\min_{\substack{U \in S^{m\times k} \\ V \in S^{k\times n}}} \left| \underset{\in\, S^{m\times n}}{A} - \underset{m\times k}{U}\ \underset{k\times n}{V} \right|_{\mathbf{0}}$$

Why $|A|_0$?

1. *Robust* to outliers/anomalies.

2. Natural for problems with *no underlying metric*.

ℓ_0-Low Rank Approximation



Binary ℓ_0-rank-k

Reals ℓ_0-rank-k

The ℓ_0-Low Rank Approximation

Classifications based on ***S***

Field

Our Result:
Poly-time Bicriteria Algorithm

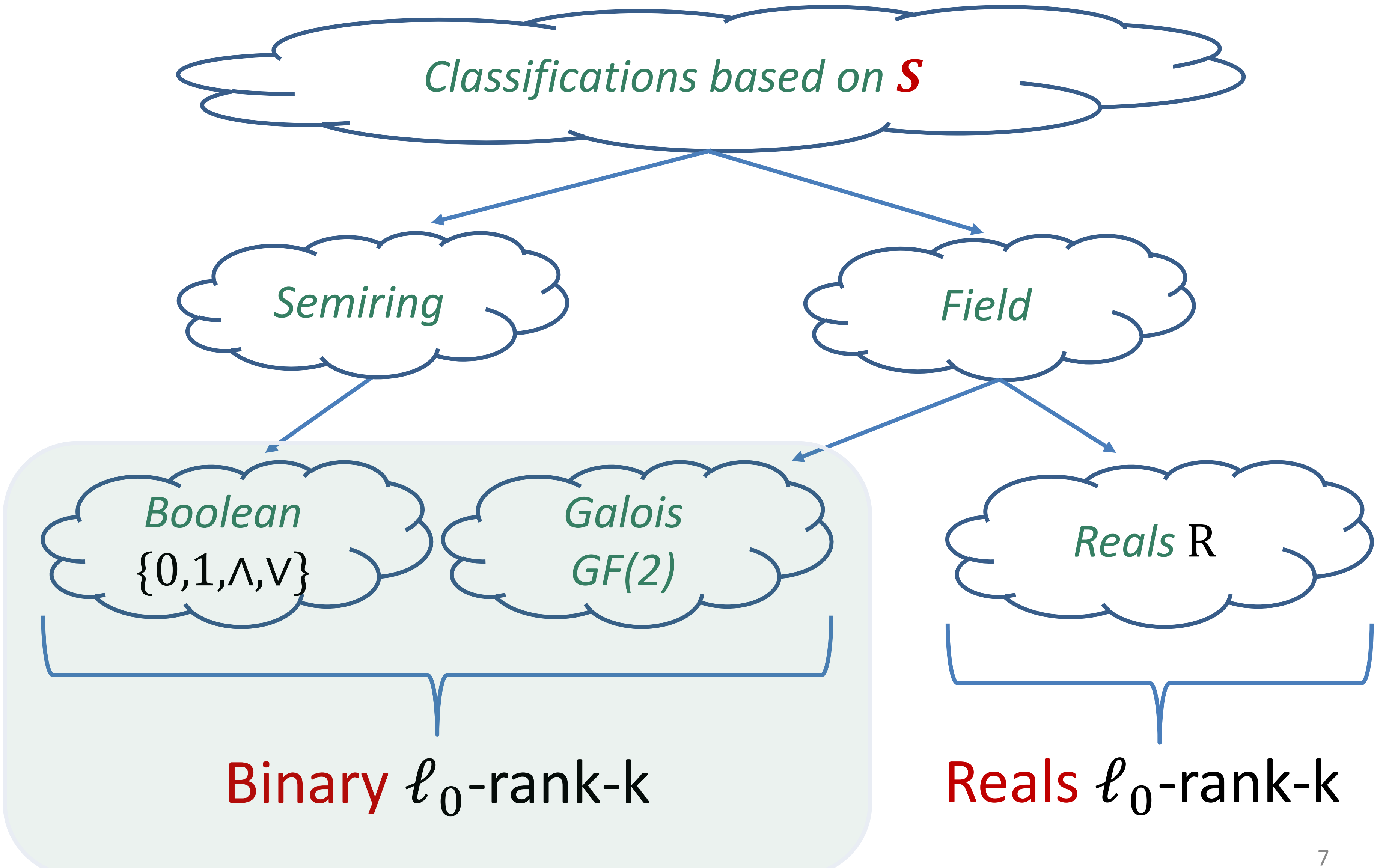
approximately solves
original Robust PCA
by *Candes et al.*

Reals $\mathbb{R}$

Binary ℓ_0-rank-k

Reals ℓ_0-rank-k

The ℓ_0-Low Rank Approximation



Binary ℓ_0-rank-k

$A \in \{0,1\}^{m\times n}$

$$\underset{U\in\{0,1\}^{m\times k},\, V\in\{0,1\}^{k\times n}}{\operatorname{argmin}} |A - U\cdot V|_0$$

Known under many different names:

Matrix Rigidity – Complexity Theory

Boolean Factor Analysis
Binary Matrix Factor – Machine Learning & Data Mining

Biclique (k-way) **Partition** – Graph Theory

NP-Hard even for $k = 1$

Binary ℓ_0-rank-1

Our Algorithmic Results

$A \in \{0,1\}^{m\times n}$

$$\mathrm{OPT} = \min_{u\in\{0,1\}^m, v\in\{0,1\}^n} |A - u\cdot v|_0 \leq |A|_0$$

Small $\phi = \frac{\mathrm{OPT}}{|A|_0}$

Exact Alg.

Runtime: $O(|A|_0 + m + n)$

Returns: $(1+\phi)$ −*Approx.*

Runtime: $2^{\mathrm{OPT}/\sqrt{|A|_0}}\mathrm{poly}(nm)$

Returns: *An **Optimal** Solution*

Good Approx: $A \approx$ *rank-1 matrix*

PolyTime: $\mathrm{OPT} \leq O(\log(nm))\sqrt{|A|_0}$

Binary ℓ_0-rank-k

$$\underset{U\in\{0,1\}^{m\times k},\, V\in\{0,1\}^{k\times n}}{\operatorname{argmin}} |A - U\cdot V|_0$$

$A \in \{0,1\}^{m\times n}$

NP-Hard even for $\boldsymbol{k} = 1$

Does the Binary ℓ_0-rank-$\boldsymbol{k}$ *admits a* ***Polynomial Time Approximation Scheme (PTAS)*** *?*

In a follow up work,
we give an **affirmative** answer.

Thank You

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