CWI Centrum Wiskunde & Informatica

Computing good Nash equilibria in combinatorial congestion games

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Congestion games

A *congestion game* Γ is given by the tuple $(N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E})$ with

- set of players N with $n = |N|$,
- set of resources E with $m = |E|$,
- strategy set $\mathcal{S}_i \subseteq 2^E$ for every $i \in N$,
- cost function c_e for every $e \in E$.

Player objective: minimize total cost over all resources used, i.e., minimize

$$C_i(s) = \sum_{e\in s_i} c_e(x_e)$$

where $s = (s_1, \dots, s_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ is a *strategy profile* with x_e the number of players using resource e in strategy profile s.

Pure Nash equilibrium: strategy profile s such that

$$C_i(s) \leq C_i(s_{-i}, s_i')$$

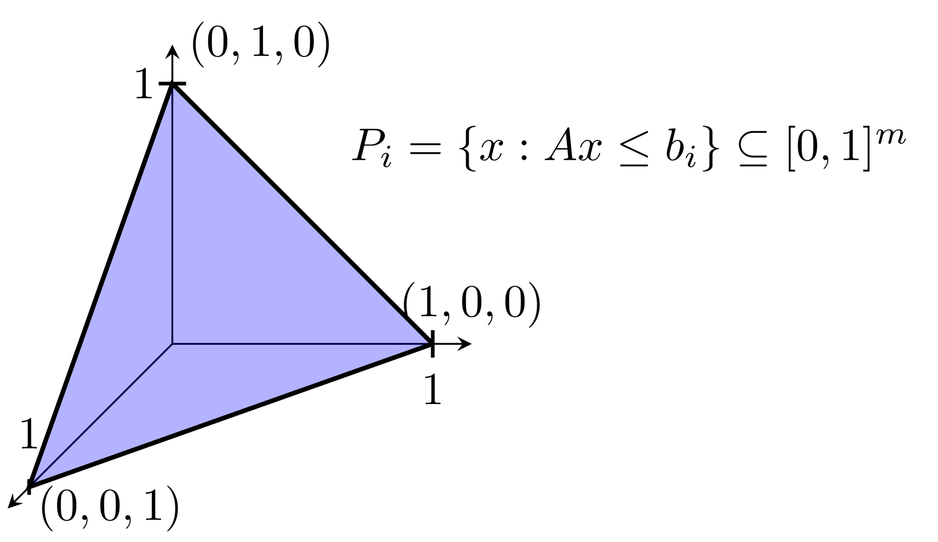
for all $i \in N$ and $s_i' \in \mathcal{S}_i$. These are the local minima of *Rosenthal's potential*:

$$\Phi(s) = \sum_{e\in E}\sum_{k=1}^{x_e} c_e(k).$$

That is, for all $i \in N$ and $s_i' \in \mathcal{S}_i$: $\Phi(s) - \Phi(s_{-i}, s_i') = C_i(s) - C_i(s_{-i}, s_i')$.

Polytopal strategy sets [Del Pia et al. (2017)]

Strategy set $\mathcal{S}_i \subseteq 2^E$ given by *extreme points of polytope* P_i for $i \in N$.



The points $\{0,1\}^m \cap P_i$ represent the **incidence vectors of strategies** in $\mathcal{S}_i$. The **aggregation polytope** P_N is defined by

$$P_N = \sum_{i\in N} P_i.$$

MAIN RESULT (informal):

Identify sufficient polytopal properties of P_N that allow for polynomial time computation of good Nash equilibria (unifying and extending existing work).

- **Integer Decomposition Property (IDP)**:

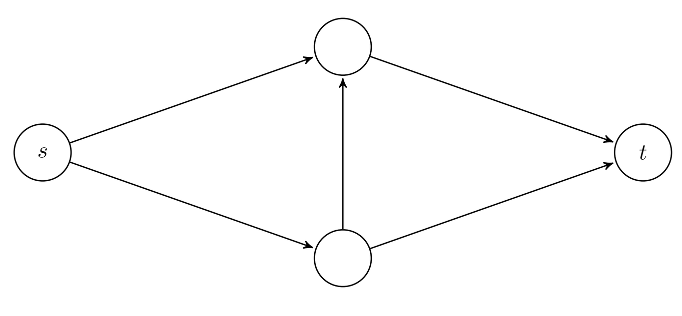
$$\forall y \in P_N \cap \{0,n\}^m \ \ \exists y_i \in P_i \cap \{0,1\}^n \text{ such that } y = \sum_i y_i$$

  - Relevance for congestion games emerges in [Del Pia, Ferris and Michini (2017)].
- **box-Total Dual Integrality (box-TDI):**
  - Technical condition sufficient to guarantee, among other things, box-integrality (intersection of integral polytope with integral box being integral).

Applications

Congestion games with strategy sets of the following forms.

- Symmetric totally unimodular (e.g., symmetric network).
- Base matroid (e.g., spanning trees in undirected graph).
- Symmetric r-arborescence (directed spanning tree rooted in r).
- Common source network.



Computing Rosenthal minimizer

Use two-step approach [Del Pia et al. (2017)]:

i) **Aggregation:** Compute a feasible load profile f^* minimizing Rosenthal's potential.
- CONTRIBUTION: Can do this if P_N has IDP + box-TDI.
- Gives rise to (strongly) polynomial time algorithms for this phase (relying on ellipsoid method).

ii) **Decomposition:** Decompose f^* into a feasible strategy profile.
- (OPEN) Can we always decompose in polynomial time?
- Known for individual applications and in case P_N satisfies (stronger) **middle integral decomposition property.**

Price of Stability

Quality of strategy profile s is measured by social cost

$$C(s) = \sum_{i\in N} C_i(s) = \sum_{e\in E} x_e c_e(x_e).$$

Price of Stability (PoS): compare best Nash equilibrium against social optimum.

$$\text{PoS}(\Gamma) = \frac{\min_{s\in \text{NE}} C(s)}{\min_{s^*\in \times_i \mathcal{S}_i} C(s^*)}$$

CONTRIBUTION:

Let P_N have **IDP + box-TDI**, then for cost functions in class $\mathcal{D}$ we have $PoS(\Gamma) \leq \rho(\mathcal{D})$.

- $\rho(\mathcal{D})$ is *price of anarchy* for non-atomic routing games [Correa et al., 2004].
  - For $\mathcal{D}$ the class of polynomials of degrees at most d:

$$\rho(\mathcal{D}) = \left(1 - \frac{d}{(d+1)^{(d+1)/d}}\right)^{-1} = \Theta\left(\frac{d}{\ln d}\right).$$

- Generalization of [Fotakis, 2010] for symmetric network case.
- Improves asymptotic bound of $d + 1$ for price of stability in general congestion games [Christodoulou and Gairing (2016)].

Bottleneck congestion games

A *bottleneck congestion game* is also given by tuple $(N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E})$ as before.

Player objective: minimize maximum cost over all resources used, i.e., minimize

$$C_i(s) = \max_{e\in s_i} c_e(x_e).$$

Strong equilibrium: *for all* $K \subseteq N$

$$C_i(s) \leq C_i(s_{-K}, s_K')$$

for at least one $i \in K$, *and all* $s_K' \in \times_{i\in K}\mathcal{S}_i$.

In [Harks, Hoefer, Klimm and Skopalik (2013)] an algorithm for computing a strong equilibrium based on a **strategy packing oracle** is given, and existence of an efficient oracle for various combinatorial problems is shown.

CONTRIBUTION: IDP + box-TDI (to some extent) sufficient for having efficient oracle.

References

[1] A. Del Pia, M. Ferris and C. Michini (SODA 2017). *Totally unimodular congestion games*

[2] D. Fotakis (TOCS 2010). *Congestion Games with Linearly Independent Paths: Convergence Time and Price of Anarchy.*

[3] T. Harks, M. Hoefer, M. Klimm and A. Skopalik (MP 2013). *Computing pure Nash and strong equilibria in bottleneck congestion games*

[*] P. Kleer and G. Schäfer (EC 2017). *Potential function minimizers in combinatorial congestion games: efficiency and computation*