A PTAS for Euclidean TSP with Hyperplane Neighborhoods

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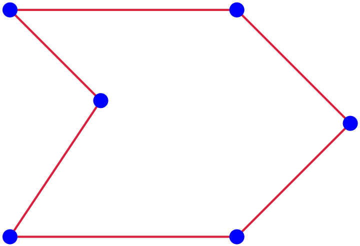
Euclidean TSP

Input: n points in $\mathbb{R}^d$.

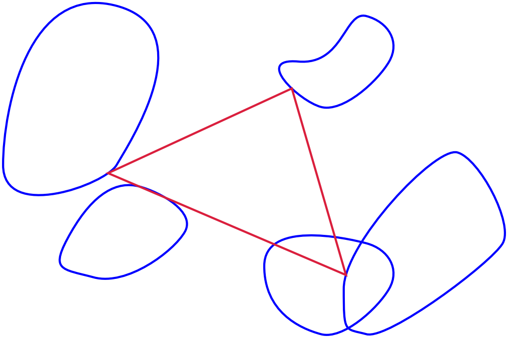
Goal: Find shortest tour visiting each point.



Euclidean TSP with neighborhoods

Input: n neighborhoods in $\mathbb{R}^d$.

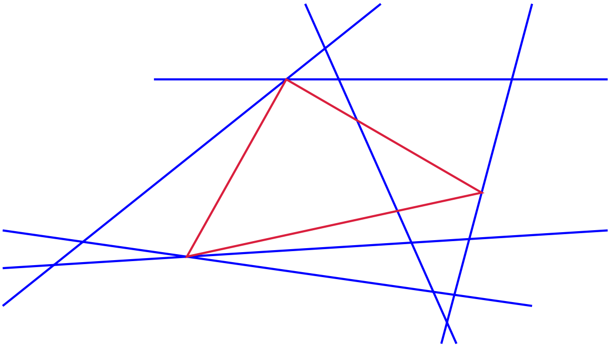
Goal: Find shortest tour visiting at least one point on each neighborhood.



Euclidean TSP with hyperplane neighborhoods

Input: n hyperplanes in $\mathbb{R}^d$, fixed d.

Goal: Find shortest tour visiting at least one point on each hyperplane.



Motivation: why hyperplanes?

- ETSP well-understood: NP-hard/admits a PTAS

  [Papadimitriou and Yannakakis, 1993; Arora, 1998; Mitchell, 1999]
- TSPN general neighborhoods: APX-hard, even in $\mathbb{R}^2$

  [Elbassioni, Fishkin and Sitters, ISAAC 2006]
- Less general neighborhoods:
  - Fat neighborhoods (radii of ball contained and ball containing differ by a constant factor): PTAS

    [Dumitrescu and Mitchell, SODA 2001; Bodlaender et al., WAOA 2006; Mitchell, SODA 2007; Chan and Elbassioni, SODA 2010; Chan and Jiang, SODA 2016]
  - Open problem: complexity hyperplane neighborhoods

    e.g. [Mitchell, SODA 2007; Dumitrescu and Tóth, SODA 2013]
  - Previous: Optimal algorithm for lines in $\mathbb{R}^2$ in $O(n^5)$ time

    [Jonsson, 2002]

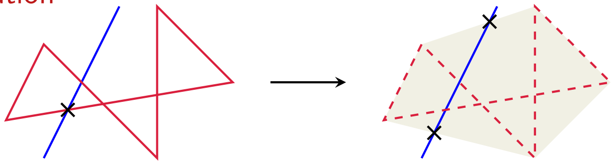
    $2^{\Theta(d)}$-approximation for hyperplanes in $\mathbb{R}^d$, $d \geq 3$

    [Dumitrescu and Tóth, SODA 2013]

Theorem

ETSP with hyperplane neighborhoods admits a PTAS.

Observation



Proof idea

Lemma Restricted polytopes are sufficient:

- Faces are parallel to $O_{\varepsilon,d}(1)$ hyperplanes.
- Proof using geometric properties.

Series of LPs approximate the optimal restricted polytope.

Come see our poster

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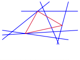
1. TSP with hyperplane neighborhoods

Input: n hyperplanes in $\mathbb{R}^d$. (Figures: $\mathbb{R}^2$)

Goal: Find shortest tour visiting at least one point on each hyperplane.



2. Related work

Arkin and Hassan [1994]: Introduction of TSP with neighborhoods problem.

Jonsson [2002]: Optimal algorithm for lines in $\mathbb{R}^2$ in $O(n^5)$.

Elbassioni, Fishkin, and Sitters [ISAAC 2006]: Similar-length line segments in $\mathbb{R}^2$ is APX-hard.

Mitchell [SODA 2007]: PTAS for "fat" neighborhoods (hyperplanes are not fat).

Dumitrescu and Tóth [SODA 2013]: $O(\log^3 n)$-approximation for lines in $\mathbb{R}^3$; $(1+\varepsilon)2^{d-1}/\sqrt{d}$-approximation for hyperplanes in $\mathbb{R}^d$, $d \geq 3$.

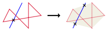
Open: Complexity for hyperplanes in $\mathbb{R}^d$.

3. Result

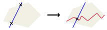
Theorem. *There is a $(1+\varepsilon)$-approximation algorithm (PTAS) for TSP with hyperplane neighborhoods (for fixed d).*

4. Observation

The convex hull of a feasible tour intersects all hyperplanes:



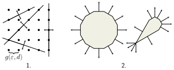
A tour of the vertices of a polytope which intersects all hyperplanes is feasible:



5. Restricted polytopes

Restrict to polytopes with few $(O_{\varepsilon,d}(1))$ faces:

1. Points from the unit cube grid define $O_{\varepsilon,d}(1)$ halfspaces (d grid points define two different halfspaces).
2. A restricted polytope is the intersection of shifted halfspaces.



6. Finding optimal restricted polytope

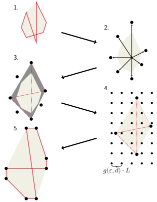
Lemma. *There is a PTAS for finding the restricted polytope (for fixed d).*

Proof (idea). Solve LPs:

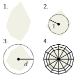
- "Guess" which halfspace boundaries intersect in vertices of the polytope.
- Variables that shift the halfspaces s.t. their boundaries intersect in the vertices.
- For each hyperplane h: "guess" a vertex-pair that is separated by h.
- "Guess" the optimal tour of the vertices.
- Restricted polytopes: $O_{\varepsilon,d}(1)$ vertices $\Rightarrow$ polynomial LP size. $\square$

7. Why restricted polytopes suffice

1. Optimal tour T and convex hull $\mathrm{Conv}(T)$.
2. Extend vertices of the convex hull by a factor of $(1+\varepsilon)$.
3. Choose $O_{\varepsilon,d}(1)$ many of the extended vertices such that their convex hull contains $\mathrm{Conv}(T)$.
4. Snap to grid points of grid on a cube with sides L (that contains $\mathrm{Conv}(T)$ exactly).
5. Restricted polytope with tour T' s.t. $\mathrm{len}(T') \leq (1+\varepsilon)\,\mathrm{len}(T)$.



8. Choosing $O(1)$ vertices



1. Polytope P.
2. $P' =$ linear transformation(P) s.t. largest volume inscribed ellipsoid is a unit ball.
3. Ball with radius d contains P' [Ball, 1992].
4. Create cells: $O_{\varepsilon,d}(1)$ rays of length d; cells with height ε. Only d vertices per ray necessary.