A PTAS for Euclidean TSP with Hyperplane Neighborhoods

Antonios Antoniadis\textsuperscript{1} \hspace{1cm} Krzysztof Fleszar\textsuperscript{2}
\textbf{Ruben Hoeksma\textsuperscript{3}} \hspace{1cm} Kevin Schewior\textsuperscript{4}

\textsuperscript{1}Saarland University & Max Planck Institute for Informatics
\textsuperscript{2}Universidad de Chile & Max Planck Institute for Informatics
\textsuperscript{3}Universität Bremen
\textsuperscript{4}École Normale Supérieure Paris & Technische Universität München

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Euclidean TSP

Input: \( n \) points in \( \mathbb{R}^d \).

Goal: Find shortest tour visiting each point.
Euclidean TSP with neighborhoods

Input: \( n \) neighborhoods in \( \mathbb{R}^d \).

Goal: Find shortest tour visiting at least one point on each neighborhood.
Euclidean TSP with hyperplane neighborhoods

Input: \( n \) hyperplanes in \( \mathbb{R}^d \), fixed \( d \).
Goal: Find shortest tour visiting at least one point on each hyperplane.
Motivation: why hyperplanes?

- ETSP well-understood: NP-hard/admits a PTAS
  [Papadimitriou and Yannakakis, 1993; Arora, 1998; Mitchell, 1999]

- TSPN general neighborhoods: APX-hard, even in $\mathbb{R}^2$
  [Elbassioni, Fishkin and Sitters, ISAAC 2006]

- Less general neighborhoods:
  - Fat neighborhoods (radii of ball contained and ball containing differ by a constant factor): PTAS
    [Dumitrescu and Mitchell, SODA 2001; Bodlaender et al., WAOA 2006; Mitchell, SODA 2007; Chan and Elbassioni, SODA 2010; Chan and Jiang, SODA 2016]

  - Open problem: complexity hyperplane neighborhoods
    e.g. [Mitchell, SODA 2007; Dumitrescu and Tóth, SODA 2013]

- Previous: Optimal algorithm for lines in $\mathbb{R}^2$ in $O(n^5)$ time
  [Jonsson, 2002]

  $2^{\Theta(d)}$-approximation for hyperplanes in $\mathbb{R}^d$, $d \geq 3$
  [Dumitrescu and Tóth, SODA 2013]
Theorem
ETSP with hyperplane neighborhoods admits a PTAS.

Observation

Proof idea

Lemma Restricted polytopes are sufficient:
- Faces are parallel to $O_{\varepsilon,d}(1)$ hyperplanes.
- Proof using geometric properties.

Series of LPs approximate the optimal restricted polytope.
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\textsuperscript{1}Saarlouis University and Max-Planck-Institut für Informatik, antonios@mpi-inf.mpg.de
\textsuperscript{2}Universidad de Chile, kleszar@hpi-uchile.cl
\textsuperscript{3}Universität Bremen, hoeksma@uni-bremen.de
\textsuperscript{4}École Normale Supérieure Paris and Technische Universität München, kechewior@gmail.com

1. TSP with hyperplane neighborhoods

Input: hyperplanes in $\mathbb{R}^d$ (Figure: $3\mathbb{R}^2$)
Goal: Find shortest tour visiting at least one point on each hyperplane.

2. Related work

Arkin and Hassan \textsuperscript{(1994)}: Introduction of TSP with neighborhoods problem.
Jonsson \textsuperscript{(2002)}: Optimal algorithm for lines in $\mathbb{R}^2$ in $O(n^2)$.
Elbassioni, Fishkin, and Sitters \textsuperscript{(ISAAC 2006)}: Similar-length line segments in $\mathbb{R}^2$ in APX-hard.
Mitchell \textsuperscript{(SODA 2007)}: PTAS for “fat” neighborhoods (hyperplanes are not fat).
Dumitrescu and Tóth \textsuperscript{(SODA 2013)}: $O(\log d)$-approximation for lines in $\mathbb{R}^2$; $(1 + \epsilon)^d \cdot O(d)$-approximation for hyperplanes in $\mathbb{R}^d$.

3. Result

Theorem. There is a $(1 + \epsilon)$-approximation algorithm (PTAS) for TSP with hyperplane neighborhoods (for fixed $d$).

4. Observation

The convex hull of a feasible tour intersects all hyperplanes.

5. Restricted polytopes

Restrict to polytopes with few $O_d(1)$ faces:
1. Points from the unit cube grid define $O_d(1)$ halfspaces
   ($d$ grid points define two different halfspaces).
2. A restricted polytope is the intersection of shifted hyperplanes.

6. Finding optimal restricted polytope

Lemma. There is a PTAS for finding the restricted polytope (for fixed $d$).
Proof (idea): Solve LPs.

7. Why restricted polytopes suffice

1. Optimal tour $T$ and convex hull $\text{Conv}(T)$.
2. Extend vertices of the convex hull by a factor of $(1 + \epsilon)$.
3. Choose $O_d(1)$ of the extended vertices such that their convex hull contains $\text{Conv}(T)$.
4. Snap to grid points of grid on a cube with sides $L$ (that contains $\text{Conv}(T)$ exactly).
5. Restricted polytopes with tour $T'$ s.t. $\text{len}(T') \leq (1 + \epsilon) \cdot \text{len}(T)$.

8. Choosing $O(1)$ vertices

1. Polytope $P$.
2. $P' = \text{linear transformation of } P$ s.t. largest volume inscribed ellipsoid is a unit ball.
3. Ball with radius $\epsilon$ contains $P'$ (Ball, 1992).
4. Create cells: $O_d(1)$ rays of length $d$; cells with height $\epsilon$. Only $d$ vertices per ray necessary.